Algebraic Multigrid for Edge Elements by Component Splitting

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Overview:

- **Low-frequency problems in electrodynamics**
- **Hybrid AMG-preconditioner:**
  - **Coping with the kernel of** $curl$
  - **Coarsening via component splitting**
- **Numerical experiments**

Maxwell's Equations

$$\begin{aligned} curl\boldsymbol{H} &= \boldsymbol{j} + \epsilon\,\partial_t\boldsymbol{E} \\ curl\boldsymbol{E} &= -\mu\,\partial_t\boldsymbol{H} \end{aligned}$$

Ohm's law: $\boldsymbol{j} = \sigma\boldsymbol{E}$

$$\boldsymbol{H},\ \boldsymbol{E} \in \boldsymbol{H}(curl)$$

(1) Stationary time-harmonic fields: $\boldsymbol{E}(\boldsymbol{x},t) = \mathrm{Re}\{\widehat{\boldsymbol{E}}(\boldsymbol{x})e^{i\omega t}\}\,,\ \ldots$

Low frequency range: $\lambda \leq \mathrm{diam}(\Omega)$

$$curl(\frac{1}{\mu}curl\widehat{\boldsymbol{E}}) - \omega^2\widehat{\epsilon}\,\widehat{\boldsymbol{E}} = \widehat{\boldsymbol{f}}\,, \qquad \widehat{\epsilon} = \epsilon - \sigma/i\omega$$

$$(\frac{1}{\mu}curl\widehat{\boldsymbol{E}},\, curl\widehat{\boldsymbol{v}}) - \omega^2(\widehat{\epsilon}\,\widehat{\boldsymbol{E}},\widehat{\boldsymbol{v}}) = (\widehat{\boldsymbol{f}},\widehat{\boldsymbol{v}}) \qquad \forall\widehat{\boldsymbol{v}} \in \boldsymbol{H}(curl)$$

(2) Transient eddy current computations:

$$\lambda \ll \mathrm{diam}(\Omega)\,, \quad \partial_t\boldsymbol{E} \to 0$$

$$curl(\frac{1}{\mu}curl\boldsymbol{E}) = \sigma\,\frac{\partial\boldsymbol{E}}{\partial t}$$

Time-stepping by backward Euler method:

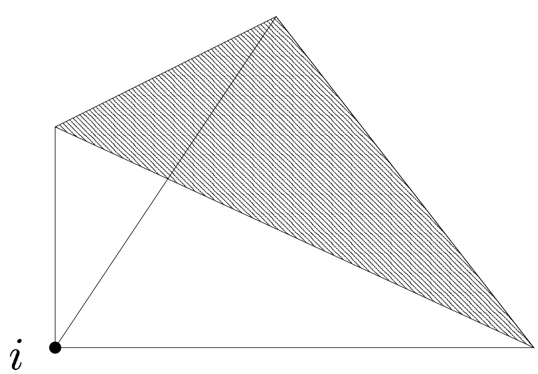
$$(\frac{1}{\mu}curl\boldsymbol{E}_n,\, curl\boldsymbol{v}) + \frac{1}{\Delta t_n}(\sigma\boldsymbol{E}_n,\boldsymbol{v}) = f(\boldsymbol{E}_{n-1},\boldsymbol{v},\ldots) \qquad \forall\boldsymbol{v} \in \boldsymbol{H}(curl)$$

Representations for Fields on Simplicial Grids: Lagrange-Type Bases and Edge Elements

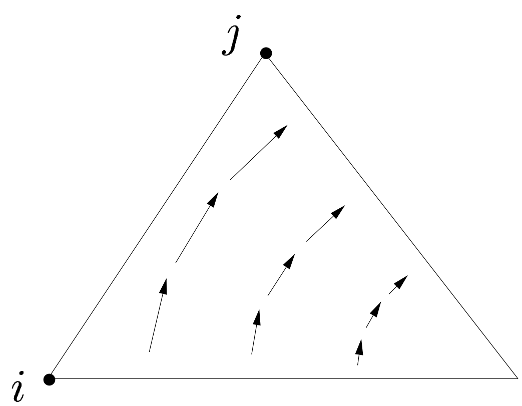
$\lambda_i(\boldsymbol{x})$: barycentric shape function for vertex i

vertices:	$\lambda_i(\boldsymbol{x})$	$\rightarrow \phi \quad \in S_h \subset H^1$ *continuous scalar* linear nodal elements
edges:	$\boldsymbol{v}_{\{ij\}} = \lambda_i \nabla \lambda_j - \lambda_j \nabla \lambda_i$	$\rightarrow \boldsymbol{E} \quad \in N_h \subset \boldsymbol{H}(curl)$ *tangential continuity* "edge elements"

2D - case:



$\lambda_i(\boldsymbol{x})$



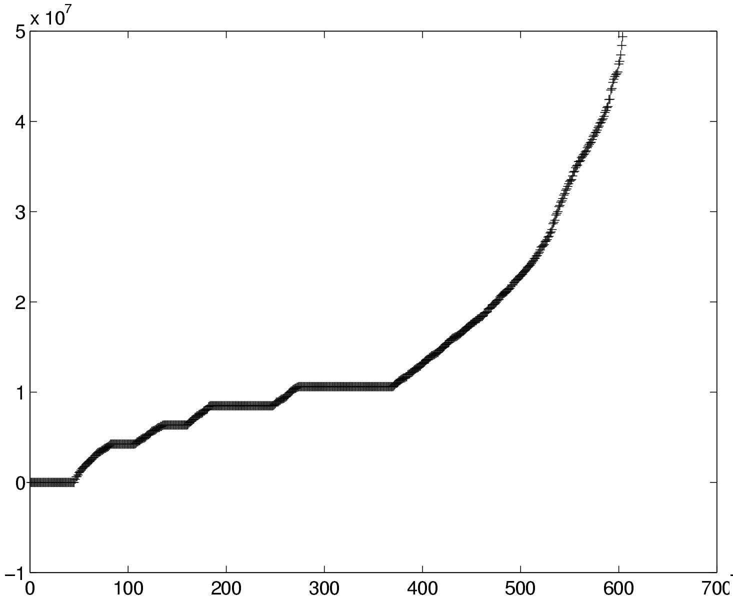
$\boldsymbol{v}_{\{ij\}} = \lambda_i \nabla \lambda_j - \lambda_j \nabla \lambda_i$

Spectral Properties of System Matrices

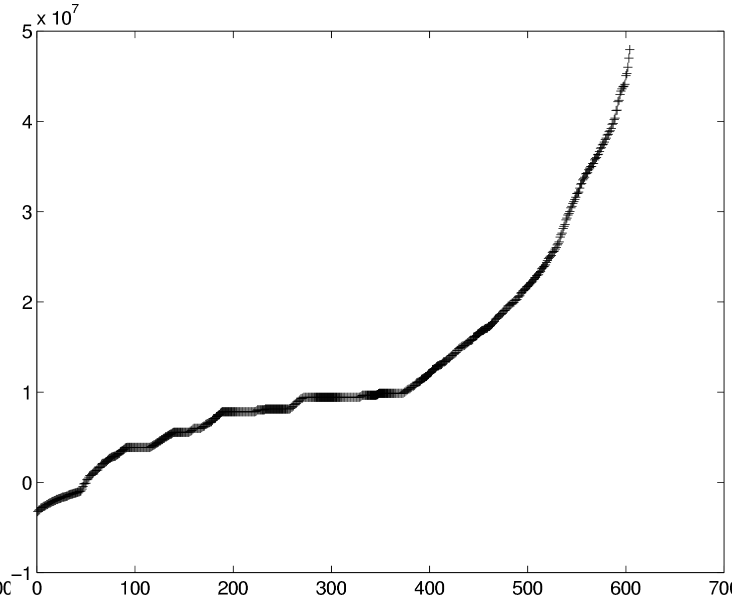
$$(\frac{1}{\mu} curl\boldsymbol{E},\, curl\boldsymbol{v}) \;-\; \omega^2(\epsilon\,\boldsymbol{E},\boldsymbol{v}) \;=\; (\widehat{\boldsymbol{f}},\widehat{\boldsymbol{v}}) \qquad\qquad \forall \boldsymbol{v} \in \boldsymbol{H}(curl)$$

$$\boldsymbol{H}(curl) \rightarrow N_h : \qquad \boldsymbol{E} \;=\; \sum_{\{ij\}} \boldsymbol{v}_{\{ij\}} x_{\{ij\}}$$

$$\rightarrow \qquad\qquad A_N x_N = r_N$$



$(\frac{1}{\mu} curl\boldsymbol{E},\, curl\boldsymbol{v})$



$(\frac{1}{\mu} curl\boldsymbol{E},\, curl\boldsymbol{v}) \;-\; \omega^2(\epsilon\,\boldsymbol{E},\boldsymbol{v})$

Helmholtz-decomposition: $\quad \boldsymbol{E} \;=\; \boldsymbol{E}_0 + \boldsymbol{E}_S\,, \qquad \boldsymbol{E} \in N_h$

$$curl\boldsymbol{E}_0 \;=\; 0 \quad \rightarrow \quad \boldsymbol{E}_0 \;\in\; grad\; S_h, \quad \boldsymbol{E}_0 \;= grad\,\phi \;=\; \sum_i \phi_i\, grad\,\lambda_i$$

Transfer operator $\quad P_{S_h} :\; S_h \rightarrow N_h \;:$

$$x_{\{ij\}} \;=\; \int_{\boldsymbol{x}_1}^{\boldsymbol{x}_2} \boldsymbol{E}_0 \cdot \mathbf{t}_{\{ij\}}\, ds \;=\; \int_{\boldsymbol{x}_1}^{\boldsymbol{x}_2} grad\,\phi \cdot \mathbf{t}_{\{ij\}}\, ds \;=\; \phi(\boldsymbol{x}_2) - \phi(\boldsymbol{x}_1)\;.$$

$\longrightarrow$ Subspace iteration in S_h for smoothing of kernel-modes:

$$A_N x_N \;=\; r_N \quad \rightarrow \quad A_\phi x_\phi \;=\; r_\phi$$

Prolongation: $\qquad x_N = P_{S_h}\, x_\phi$

Canonical Restriction: $\qquad \begin{cases} A_\phi = P_{S_h}^T\, A_N\, P_{S_h} \\ r_\phi = P_{S_h}^T\, r_N \end{cases}$

Preconditioner for $A_N x_N \;=\; r_N$ with hybrid smoothing:

(1) $\quad x_N \leftarrow 0,\;\; x_\phi \leftarrow 0$

(2) $\quad$ One forward GS-sweep on $A_\phi x_\phi = P_{S_h}^T\, r_N$

(3) $\quad x_N \leftarrow x_N + P_{S_h} x_\phi$

(4) $\quad$ One symmetric GS-sweep on $A_N x_N = r_N$

(5) $\quad x_\phi \leftarrow 0$

(6) $\quad$ One backward GS-sweep on $A_\phi x_\phi = P_{S_h}^T\,(r_N - A_N x_N)$

(7) $\quad x_N \leftarrow x_N + P_{S_h} x_\phi$

Graph-Based Coarsening Scheme for AMG

$$\text{Matrix graph of } A_\phi \quad \longleftrightarrow \quad \text{FE-grid}$$

Coarsening rules:

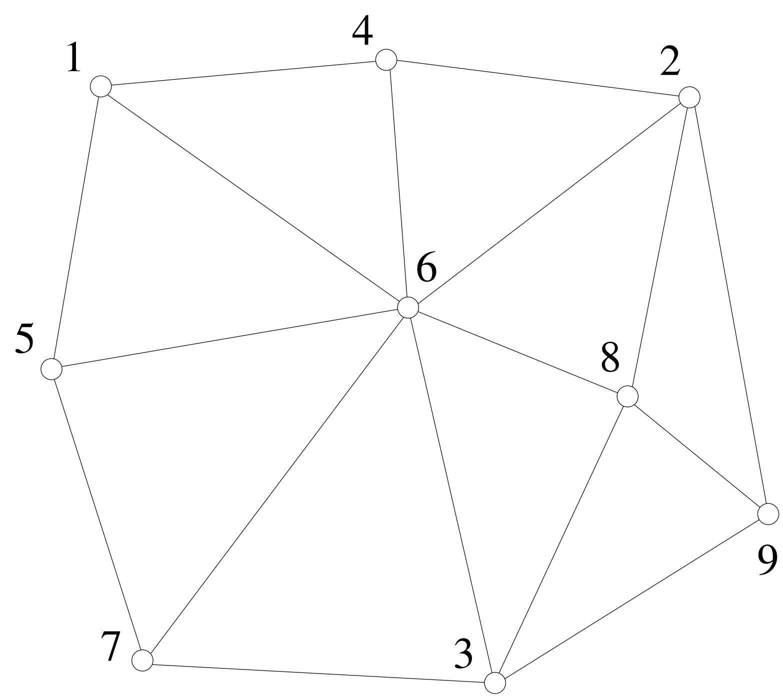
(1) **"Coarse grid nodes"**: will be kept on the coarse grid, weight = 1.0

(2) **"Fine grid nodes"** : will be dropped, interpol. weight = $1.0/n$
(n : number of coarse grid nodes the fine grid node is connected to)

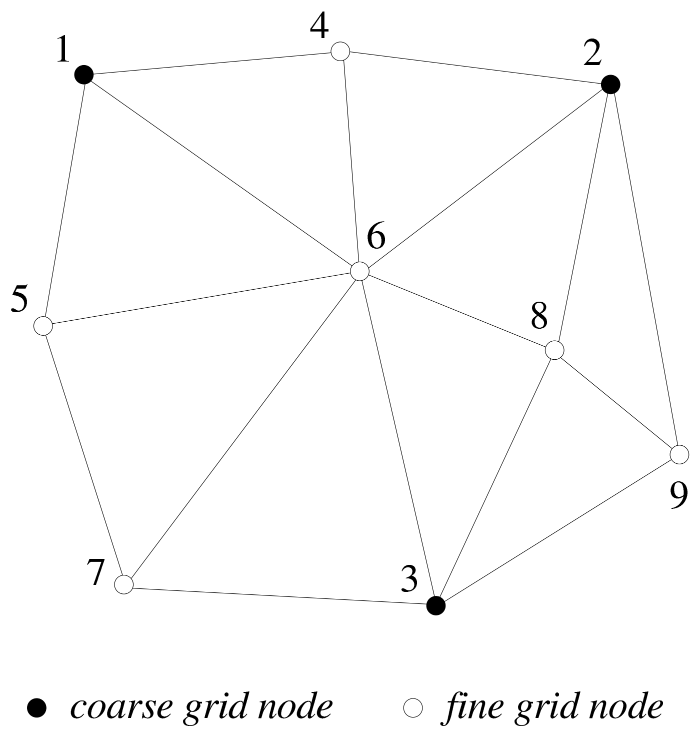
(3) No coarse grid node may be directly connected to another coarse grid node

(4) Insert as many coarse grid nodes as possible

Determination of Coarse Grid Nodes by Advancing-Front Algorithm



Example



$$P = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ \frac{1}{2} & \frac{1}{2} & 0 \\ 1 & 0 & 0 \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ 0 & 0 & 1 \\ 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & \frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

$$x_i^{fine} = \sum_k P_{ik}\, x_k^{coarse}$$

Coarsening of A_N ?

'Algebraic' viewpoint:

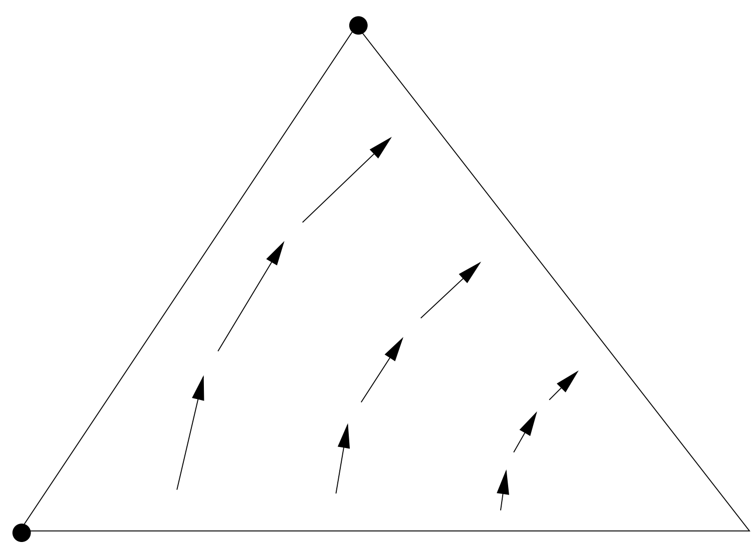
- A_N is no M-matrix, not diagonally dominant
- Large number of kernel modes (negative eigenvalues)

'Geometric' viewpoint:

- FE-basis: 'crooked' vector fields
- To what extent does aggregation

$$\boldsymbol{v}_K^{coarse} = \sum_k R_{Km}\, \boldsymbol{v}_m^{fine}$$

make sense?



Non-Conforming Nodal Basis for Vector Fields

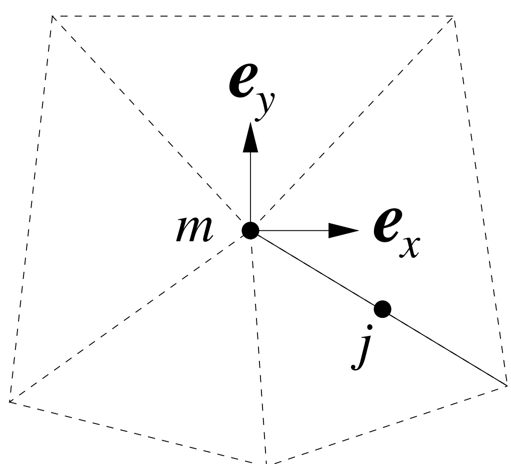
$$curl(\frac{1}{\mu} curl \boldsymbol{E}) \, - \, \omega^2 \, \epsilon \, \boldsymbol{E} \, = \, 0$$

$$-\frac{1}{\mu} \Delta \boldsymbol{E} \, + \, \frac{1}{\mu} grad(div \boldsymbol{E}) \, - \, \omega^2 \, \epsilon \, \boldsymbol{E} \, = \, 0$$

Cartesian coordinate frame: $\widetilde{\boldsymbol{E}}(\boldsymbol{x}) \, = \, \tilde{E}_x(\boldsymbol{x}) \cdot \boldsymbol{e}_x \, + \, \tilde{E}_y(\boldsymbol{x}) \cdot \boldsymbol{e}_y$

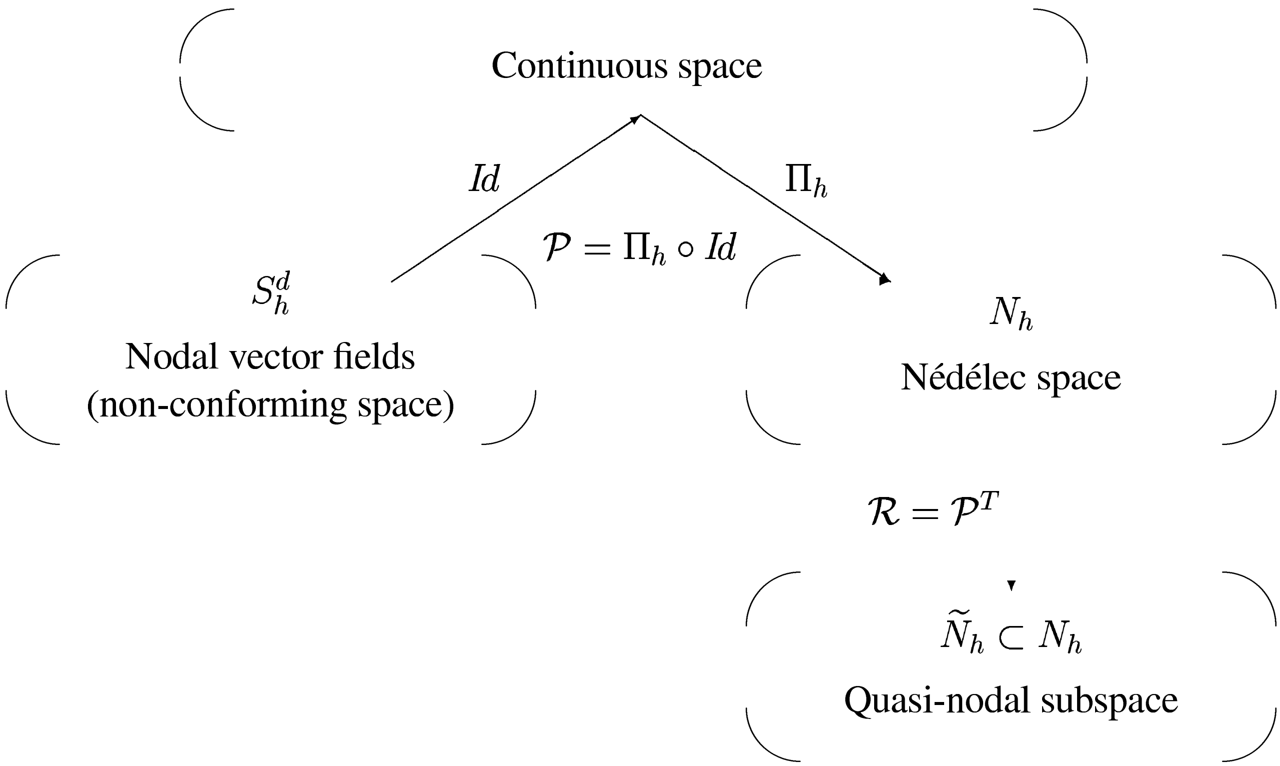
$$\tilde{E}_x(\boldsymbol{x}) \, = \, \sum_k \tilde{E}_{x,k} \, \lambda_k(\boldsymbol{x})$$

$$\tilde{E}_y(\boldsymbol{x}) \, = \, \sum_l \tilde{E}_{y,l} \, \lambda_l(\boldsymbol{x})$$



$$\mathcal{P}_{jk} \, = \, \int_{e_j} \lambda_m(\boldsymbol{x}) \, \boldsymbol{e}_i \cdot \mathbf{t}_j \, ds \; = \, \frac{1}{2} \boldsymbol{e}_i \cdot \mathbf{t}_j \, l_j \, .$$

Auxiliary Vertex-Based Spaces $\leftrightarrow$ Nédélec space



$\widetilde{N}_h$ spanned by $\quad \tilde{\boldsymbol{v}}_l \;=\; \mathcal{P}^T_{lm}\, \boldsymbol{v}_m$

Structure of Auxiliary Matrices

Quasi-nodal subspace $\widetilde{N}_h$ **:** $\quad A_{\tilde{N}} = \mathcal{R} A_N \mathcal{P}$

$$A_{\tilde{N}} = \begin{pmatrix} A_{\tilde{N}}{}^{xx} & A_{\tilde{N}}{}^{xy} & A_{\tilde{N}}{}^{xz} \\ A_{\tilde{N}}{}^{yx} & A_{\tilde{N}}{}^{yy} & A_{\tilde{N}}{}^{yz} \\ A_{\tilde{N}}{}^{zx} & A_{\tilde{N}}{}^{zy} & A_{\tilde{N}}{}^{zz} \end{pmatrix}$$

Non-conforming space S_h^3 :

$$A_{S^3} = \begin{pmatrix} A_{S^3}^{xx} & 0 & 0 \\ 0 & A_{S^3}^{yy} & 0 \\ 0 & 0 & A_{S^3}^{zz} \end{pmatrix}$$

$$A_{S^3}^{xx} = A_{S^3}^{yy} = A_{S^3}^{zz}$$

$x-$, y-, and z- components of $\widetilde{N}_h$ and S_h^3 are coarsened like S_h (i.e. like a scalar field)

AMG-Preconditioner for an Approximate Solution of $A_N x_N = r_N$:

(1) $x_N \leftarrow 0,\ x_\phi \leftarrow 0$

(2) One AMG-V-cycle on $A_\phi x_\phi = P_{S_h}^T r_N$

(3) $x_N \leftarrow x_N + P_{S_h} x_\phi$

(4) One forward GS-sweep on $A_N x_N = r_N$

(5) $x_{\tilde{N}} \leftarrow 0$

(6) One AMG-V-cycle on $A_{\tilde{N}} x_{\tilde{N}} = \mathcal{P}^T (r_N - A_N x_N)$

(7) $x_N \leftarrow x_N + \mathcal{P} x_{\tilde{N}}$

(8) One backward GS-sweep on $A_N x_N = r_N$

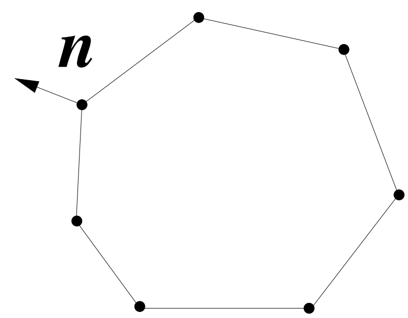
(9) $x_\phi \leftarrow 0$

(10) One AMG-V-cycle on $A_\phi x_\phi = P_{S_h}^T (r_N - A_N x_N)$

(11) $x_N \leftarrow x_N + P_{S_h} x_\phi$

Omit smoothing on top level in $\widetilde{N}_h$!

Handling of Dirichlet Boundaries in $\widetilde{\boldsymbol{N}}_h$?



$\rightarrow$ constrain *all* components belonging to vertices on Dirichlet boundaries

Shift of Indefinite Operators in $\widetilde{\boldsymbol{N}}_h$

$$(\frac{1}{\mu} curl \boldsymbol{E},\, curl \boldsymbol{v}) \;-\; \omega^2(\epsilon\, \boldsymbol{E}, \boldsymbol{v})$$

$$\rightarrow \qquad (\frac{1}{\mu} curl \boldsymbol{E},\, curl \boldsymbol{v}) \;+\; \omega^2(\epsilon\, \boldsymbol{E}, \boldsymbol{v})$$

$$A_{\tilde{N}} \;=\; \mathcal{P}^T A_N^+ \;\mathcal{P}$$

Numerical Experiments

Cojugate Gradient Solver with 4 Preconditioners:

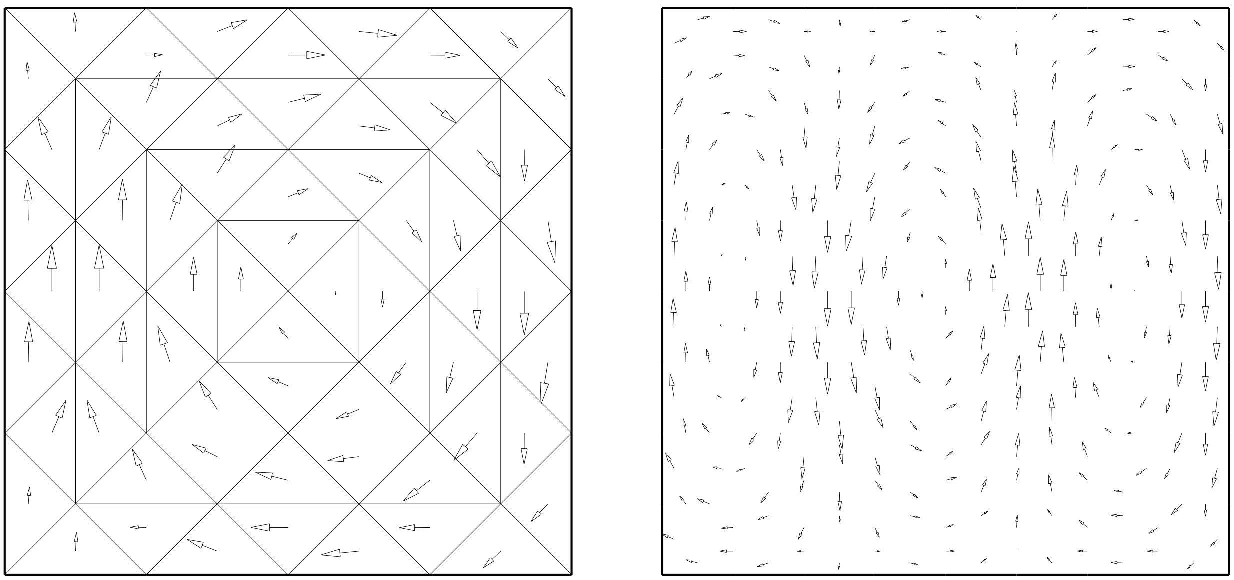
SGS:	Symmetric Gauss-Seidel in S_h and N_h
MG-SGS:	Geometric multigrid for S_h and $\widetilde{N}_h$
AMG:	Algebraic multigrid for S_h and $\widetilde{N}_h$
AMG-NC:	Algebraic multigrid for S_h and S_h^d (non-conforming)

Initial guess for solution: $x_N = 0$

Termination criterion: $|r_N^{(n)}| < 10^{-10} \, |r_N^{(0)}|$

Homogeneous 2D-Domain, Uniform Grid

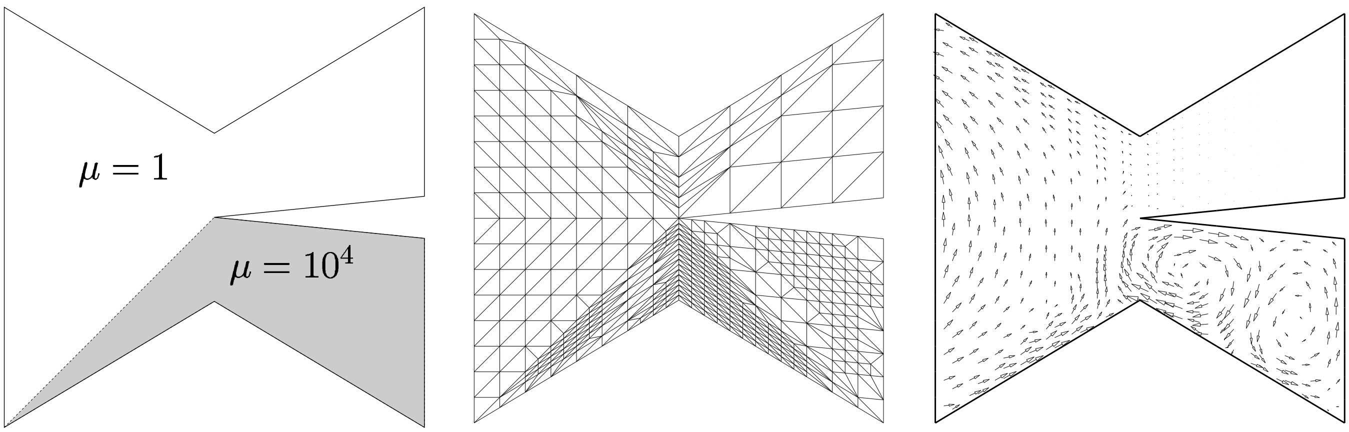
$$(\frac{1}{\mu} curl\boldsymbol{E},\, curl\boldsymbol{v}) \;-\; \omega^2(\epsilon\,\boldsymbol{E}, \boldsymbol{v}) \;=\; 0 \qquad\qquad \forall \boldsymbol{v} \in \boldsymbol{H}(curl)$$



$\dim N_h$	Iterations				CPU [sec]			
	SGS	MG-SGS	AMG	AMG-NC	SGS	MG-SGS	AMG	AMG-NC
				$\omega = 1.5\pi$				
6 176	215	14	20	25	5	0.6	1.2	1.5
24 640	424	14	21	30	45	3	6	7
98 432	840	14	20	29	414	13	23	33
393 472	1632	13	21	33	3280	48	102	154
				$\omega = 3\pi$				
6 176	351	35	42	48	8	1.6	2.4	2.6
24 640	673	32	44	50	73	6.5	12	13
98 432	1702	31	45	55	835	28	52	62
393 472	2837	33	46	56	5670	122	222	262

Inhomogeneous 2D-Domain, $\Delta\mu = 10^4$

$$\left(\frac{1}{\mu} curl\boldsymbol{E},\, curl\boldsymbol{v}\right) \;-\; \omega^2(\epsilon\,\boldsymbol{E},\boldsymbol{v}) \;=\; 0 \qquad\qquad \forall\boldsymbol{v} \in \boldsymbol{H}(curl)$$



$\dim N_h$	Iterations				CPU [sec]			
	SGS	MG-SGS	AMG	AMG-NC	SGS	MG-SGS	AMG	AMG-NC
				Uniform mesh refinement				
12 384	717	45	27	34	37	4	4	4
49 344	2478	45	30	39	577	19	18	22
196 992	$>$ 10 000	43	30	40		78	77	97
				Adaptive mesh refinement				
6 251	526	49	43	48	12	4	3	3
14 687	908	44	45	55	55	10	8	9
28 253	1694	44	49	58	217	24	17	19
64 099	2419	43	51	79	772	63	44	63
121 666	5317	42	61	87	3918	138	103	137

Homogeneous Cube, Uniform Grid

$$\left(\frac{1}{\mu} curl \boldsymbol{E},\, curl \boldsymbol{v}\right) \;-\; \omega^2(\epsilon\, \boldsymbol{E}, \boldsymbol{v}) \;=\; 0 \qquad\qquad \forall \boldsymbol{v} \in \boldsymbol{H}(curl)$$

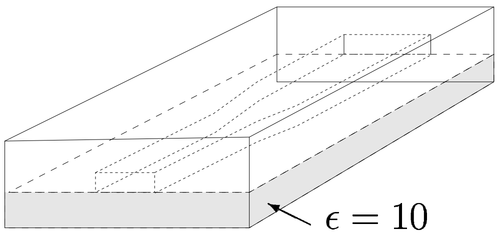
dimN_h	Iterations				CPU [sec]			
	SGS	MG-SGS	AMG	AMG-NC	SGS	MG-SGS	AMG	AMG-NC
				$\omega = 0.5\pi$				
3 976	83	25	26	27	1.2	0.6	0.7	0.7
30 224	159	24	27	32	23	6.4	7.2	8.2
235 552	303	24	32	36	460	60	90	98
1 859 648	649	24	33	38	8183	510	810	866

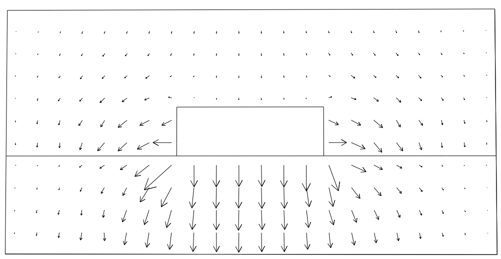
Cube, $\Delta\mu = 10^4$, Locally Refined Grid

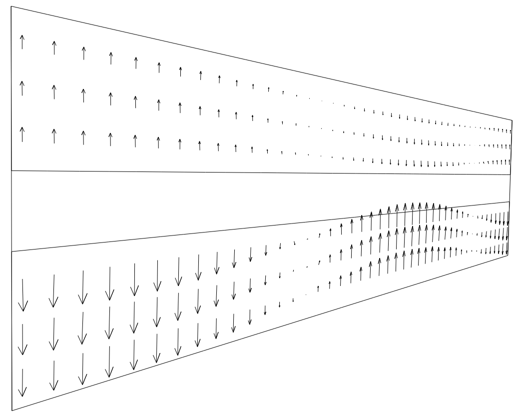
dimN_h	Iterations				CPU [sec]			
	SGS	MG-SGS	AMG	AMG-NC	SGS	MG-SGS	AMG	AMG-NC
				$\omega = 0.01\pi$				
5 445	94	19	35	37	2.0	0.9	1.3	1.3
20 190	136	19	40	41	13	5.4	7.0	6.6
100 637	210	22	41	46	130	43	45	48
371 028	348	21	43	50	893	188	205	225
1 318 273	511	21	44	53	4806	831	793	897

3D Microstrip Line

3D-Domain, $\Delta\epsilon = 10$, Locally Refined Grid







$\dim N_h$	Iterations				CPU [sec]			
	SGS	MG-SGS	AMG	AMG-NC	SGS	MG-SGS	AMG	AMG-NC
				Uniform mesh refinement				
5 396	96	45	48	61	1.8	1.3	1.4	1.8
48 424	188	44	44	68	48	19	19	28
408 656	359	44	45	75	966	188	222	340
3 354 784	692	44	49	90	15 446	1610	2097	3586
				Adaptive mesh refinement				
22 129	165	46	52	71	18	7.5	9	11
93 108	240	43	51	74	134	52	50	68
387 715	359	43	51	79	950	282	242	351
1 593 011	538	44	54	88	6272	1365	1168	1786

Summary

- Computational cost per iteration for AMG:
  $\approx$ twice as much as for one single-grid iteration
- Computation time for linear solver:
  factor 1 ... 2 compared to geometric multigrid