Element-Free AMGe: General algorithms for computing interpolation weights in AMG

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International Workshop on Algebraic Multigrid
Sankt Wolfgang, Austria
June 27, 2000

Overview

This work was performed under the auspices of the U. S. Department of Energy by the University of California Lawrence Livermore National Laboratory under contract number: W-7405-Eng-48. UCRL-VG-138290

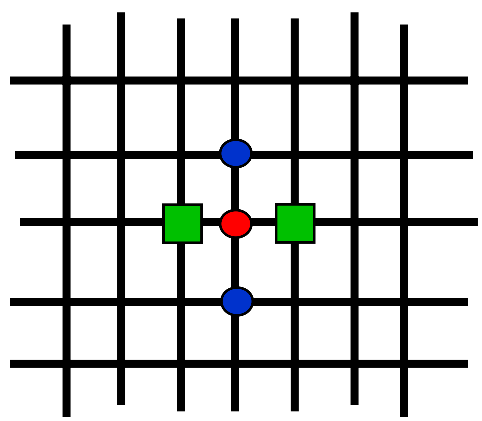
- **AMG and AMGe**
- **Element-Free AMGe: an interpolation rule based on neighborhood extensions**
- **Examples of extensions:**
  - A**-extension,**
  - L_2**-extension**
  - **extensions from minimizing quadratic functionals**
- **Numerical experiments**

AMG and AMGe

- **Assume a given sparse matrix** A
- ***AMG,* or *Algebraic Multigrid* is MG based only on the matrix entries.**
- **Essential components of AMG:**
  - **a set,** D**, of fine-grid degrees of freedom (dofs)**
  - **a coarse grid,** D_C**; typically a subset of** D
  - **a prolongation operator** $P : D_C \Rightarrow D$
  - **smoothing iterations; typically Gauß-Seidel or Jacobi**
  - **a coarse matrix given by** $A_C = P^T A P$

Coarse-grid selection

- There are several ways to select the coarse grid
- D_C is typically a maximal independent set
- each fine-grid dof is typically interpolated from a subset of its coarse nearest neighbors



Building the prolongation, *P*

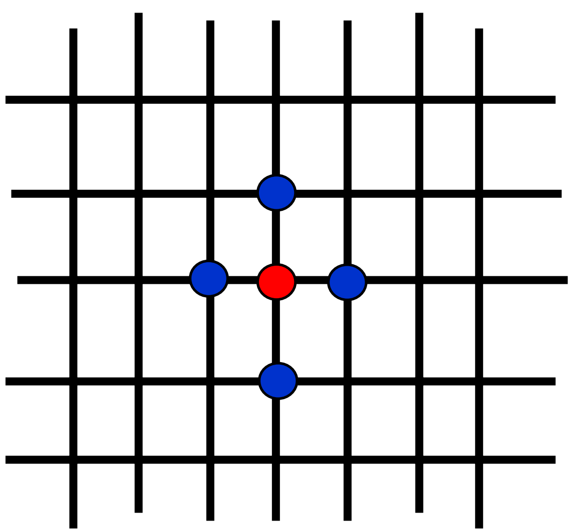
- Let $i \in D$ be a fine-grid dof
- Let $\Omega(i) \subset D$ be a neighborhood of i
- Let $\Omega_C(i)$ be the coarse-grid dofs used to interpolate a value at i
- Examine rows of A corresponding to $\Omega(i) \setminus \Omega_C(i)$.

$$A = \begin{bmatrix} A_{ff} & A_{fc} & * \\ * & * & * \\ * & * & * \end{bmatrix} \begin{matrix} \} & \Omega(i) \setminus \Omega_C(i) \\ \} & \Omega_C(i) \\ \} & D \setminus \Omega(i) \end{matrix}$$

Prolongation in classical AMG

- $\Omega(i)$ is the minimal neighborhood of i
- Replace A_{ff} with modified version, $\widehat{A}_{ff}$
  - by adding to a_{ii} all off-diagonal entries in *i* th row that are weakly connected to i
  - second, in all rows *j* for dofs strongly connected to *i* :
    - set $$a_{jj} \Leftarrow \sum_{j_c \in \Omega_c(i)} a_{jj_c}$$
    - set off-diagonals to zero
- *i*th row of *P* is *i*th row of

$$-\left(\widehat{A}_{ff}^{-1} A_{fc} \right)$$



AMGe differs from standard AMG by using finite element information

- **Traditional AMG uses the following heuristic** (based on M-matrices)**:** smooth error varies slowest in the direction of “large” coefficients

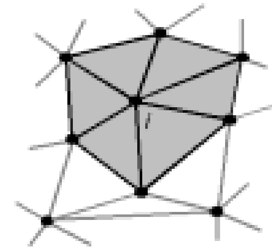
- **New heuristic based on multigrid theory:** interpolation must be able to reproduce a mode with error proportional to the size of the associated eigenvalue

- **Key idea of AMGe:** We can use information carried in the element stiffness matrices to determine
  - the nature of the smooth error components
  - accurate interpolation operators
  - the selection of the coarse grids

AMGe uses finite element stiffness matrices to localize the new heuristic

- **Local measure:**

$$M_i = \max_{e \neq 0} \frac{\left\langle \varepsilon_i \varepsilon_i^T (I - Q)\, e,\ \varepsilon_i \varepsilon_i^T (I - Q)\, e \right\rangle}{\left\langle A_i e, e \right\rangle}$$



$$A_i = \sum_{k \in \eta_i} B_k$$

where ε_i are canonical basis vectors, and A_i are sums of local stiffness matrices

- **If all local measures are small, the global measure is bounded and small $\Longrightarrow$ good convergence!**

- **Then solving a small local problem yields a row of the optimal interpolation (for the given set of C-points).**

AMGe uses a small local problem to define prolongation

- **We can show that (for a given set of interpolation points), the "optimal" prolongation is the set of weights Q that satisfy the min/max problem:**

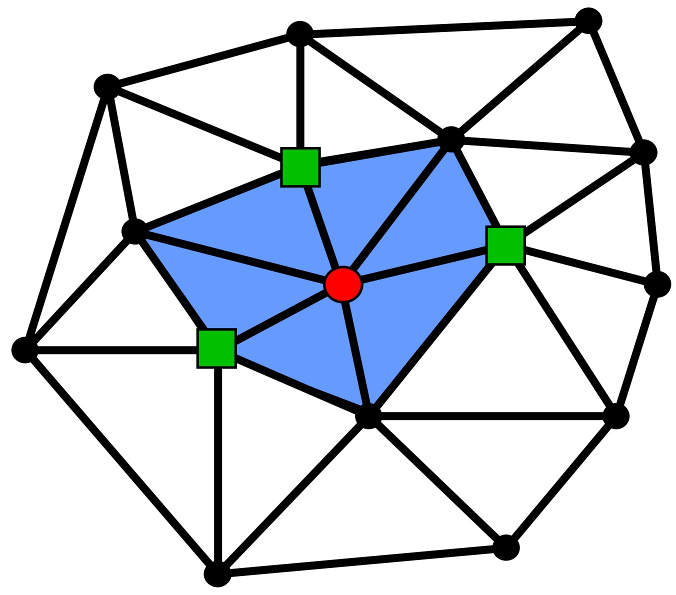
$$\min_{Q} \max_{e \neq 0} \frac{\left\langle \varepsilon_i \varepsilon_i^T (I - Q)\, e,\ \varepsilon_i \varepsilon_i^T (I - Q)\, e \right\rangle}{\left\langle A_i e, e \right\rangle}$$

- **Furthermore, solving the min/max problem is exactly equivalent to the following small matrix formulation**

Prolongation in AMGe

- The neighborhood $\Omega(i)$ is the union of the elements having i as vertex
- We use the assembled neighborhood matrix $A_{\Omega(i)}$
- Partition the neighborhood matrix as before

$$A_{\Omega(i)} = \begin{bmatrix} A_{ff} & A_{fc} \\ * & * \end{bmatrix} \begin{matrix} \} \ \Omega(i) \setminus \Omega_C(i) \\ \} \ \Omega_C \end{matrix}$$



- ith row of P is ith row of

$$-\left(A_{ff}^{-1} A_{fc}\right)$$

Prolongation in AMGe, cont.

- **Note that there is no need to modify A_{ff}**
- **Knowledge of the element matrices (used to create the assembled neighborhood matrix) carries with it implicitly the correct assignment and treatment of "weak" and "strong" connection. This is the main contribution of AMGe methods**
- **AMGe produces superior prolongation. The goal of this work is to accomplish the superior prolongation without the knowledge of the element matrices**

AMGe - Richardson vs Gauß-Seidel

	Two-level		Multilevel	
height	**amg**	**amge**	**amg**	**amge**
1	**0.97**	**0.49**	**0.98**	**0.65**
1/4	**0.97**	**0.48**	**0.98**	**0.68**
1/8	**0.98**	**0.47**	**0.99**	**0.64**
1/16	**0.97**	**0.49**	**0.99**	**0.58**
1/32	**0.97**	**0.45**	**0.98**	**0.51**
1/64	**0.98**	**0.39**	**0.98**	**0.39**

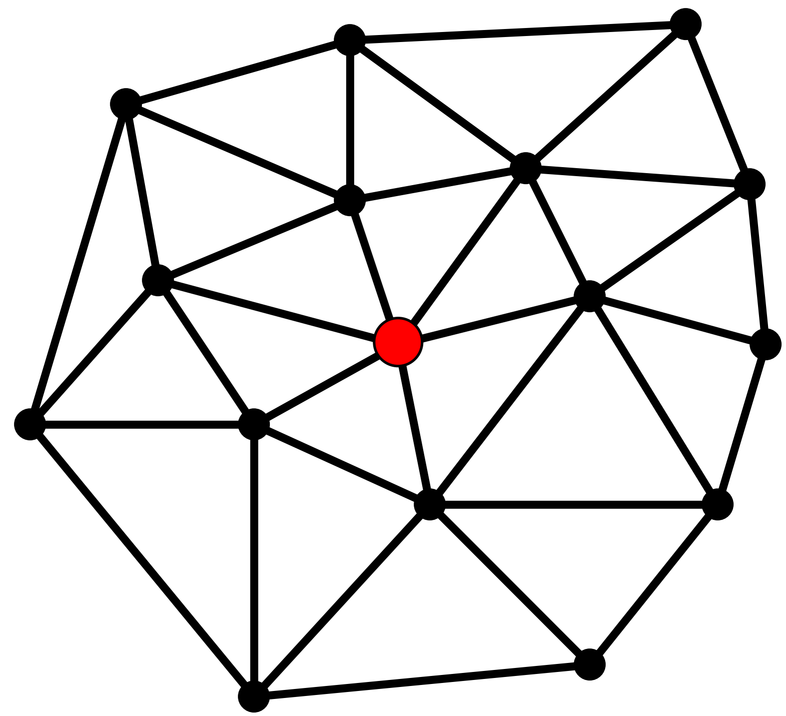
Richardson (1,0) cycle

d	*h*	AMG interpolation	AMGe interpolation
1	1/32	0.60	0.20
1/4	1/8	0.95	0.25
1/8	1/16	0.90	0.26
1/16	1/64	0.92	0.26

Gauß-Seidel (1,0) cycle

Prolongation in Element-free AMGe: based on extensions

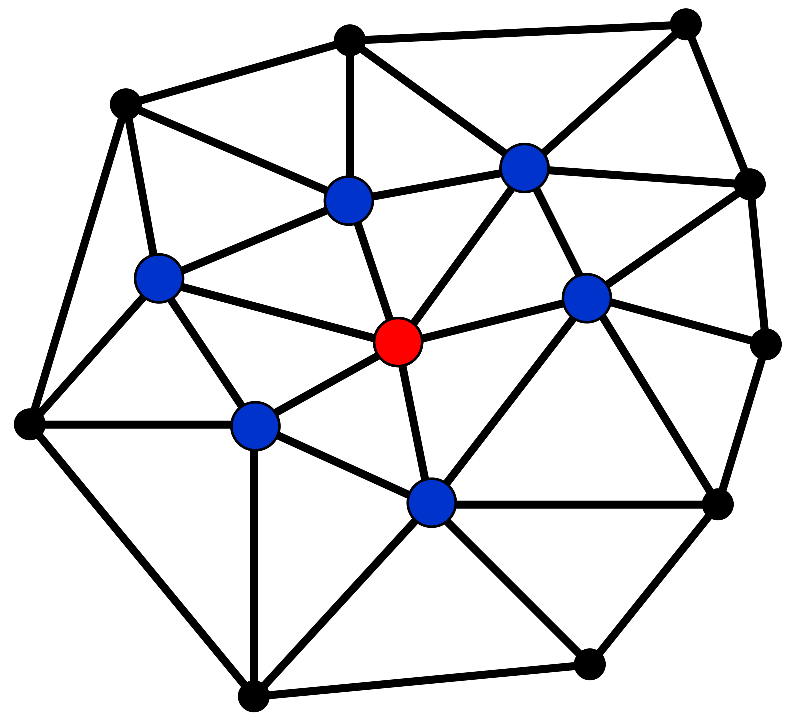
- **Let i be the f-point to which we wish to interpolate**



Prolongation in Element-free AMGe: based on extensions

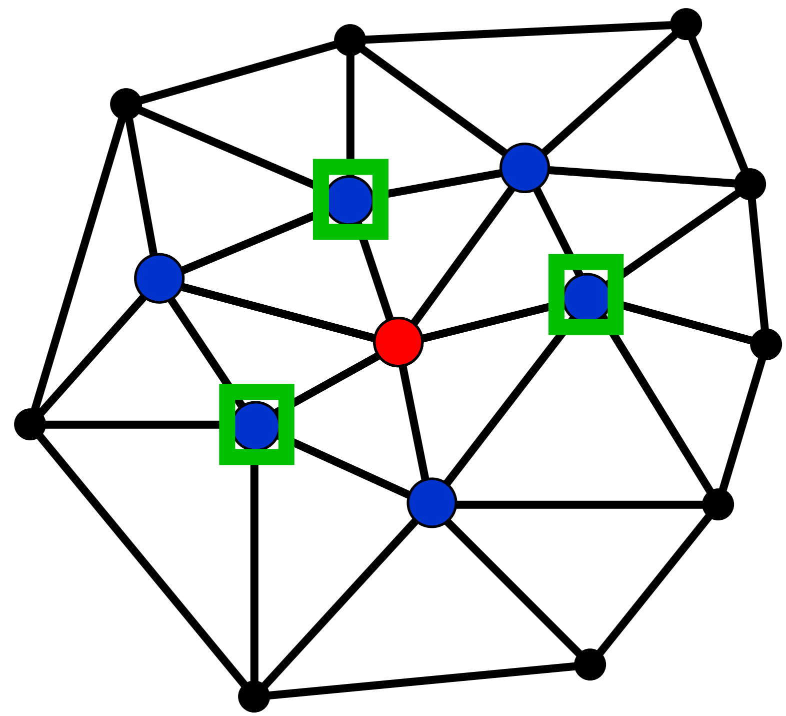
- Let i be the f-point to which we wish to interpolate

- $\Omega(i)$ is the set of points in the neighborhood of i



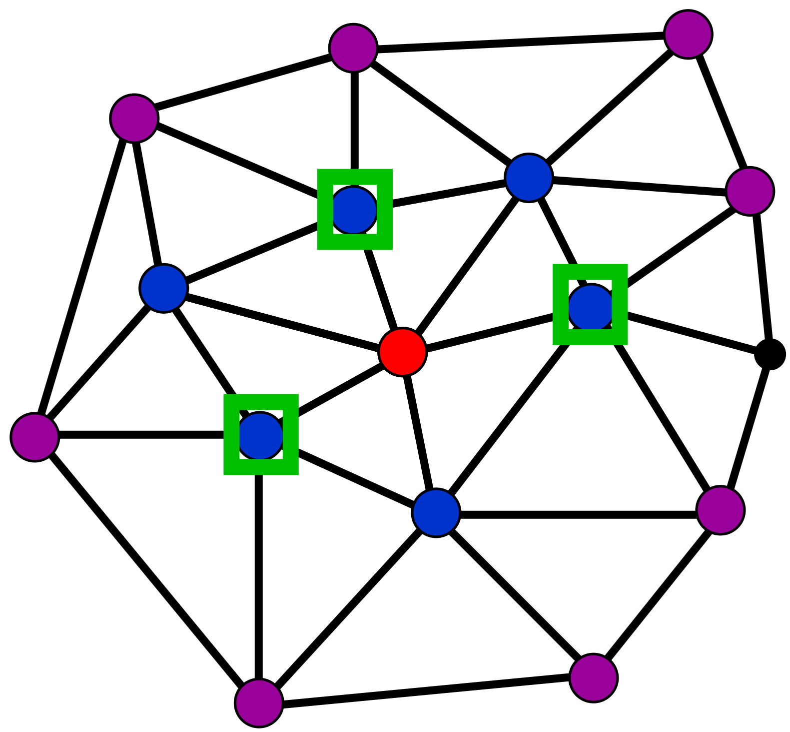
Prolongation in Element-free AMGe: based on extensions

- Let i be the f-point to which we wish to interpolate
- $\Omega(i)$ is the set of points in the neighborhood of i
- $\Omega_C(i)$ is the set of coarse nearest neighbors of i



Prolongation in Element-free AMGe: based on extensions

- **Define $\Omega_X(i)$, the set of "exterior" points for the neighborhood of i: the set of points j such that j is connected to a fine point in the neighborhood of i**



$$\Omega_X(i) = \{ j \not\subset \Omega(i) : a_{jk} \neq 0 , \; j \in \Omega(i) \setminus \Omega_C(i) \}$$

Prolongation in Element-free AMGe: based on extensions

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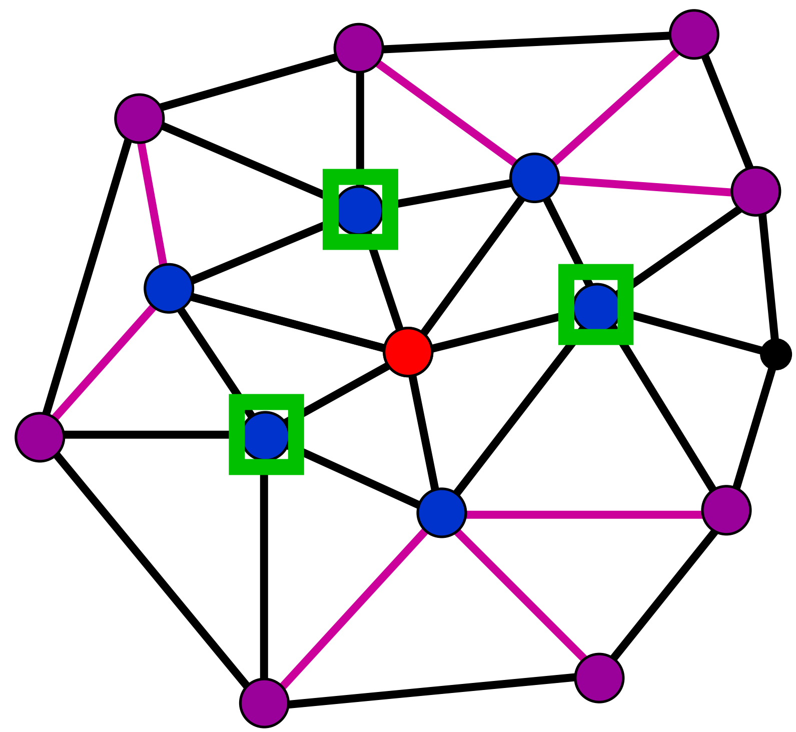


$$\Omega_X(i) = \{ j \not\subset \Omega(i) : a_{jk} \neq 0, \; j \in \Omega(i) \setminus \Omega_C(i) \}$$

Prolongation in Element-free AMGe: based on extensions

- **We use the following window of the matrix A**

$$\begin{bmatrix} A_{ff} & A_{fc} & A_{fX} & 0 \\ * & * & * & * \\ A_{Xf} & A_{Xc} & A_{XX} & * \\ * & * & * & * \end{bmatrix} \begin{matrix} \Omega(i) \setminus \Omega_c(i) \\ \Omega_c(i) \\ \Omega_X(i) \\ \text{everything else on grid} \end{matrix}$$

where we will only be interested in the blocks shown.

Prolongation in Element-free AMGe: based on extensions

- **Assume that an extension mapping is available:**

$$E = \begin{bmatrix} I & 0 \\ 0 & I \\ E_{Xf} & E_{Xc} \end{bmatrix}$$

i.e., we interpolate the exterior dofs ("*X*") from the interior dofs *f* and *c*, by the rule

$$v_X = E_{Xf} v_f + E_{Xc} v_c$$

Prolongation in Element-free AMGe: based on extensions

- **We construct the prolongation operator on the basis of the modified matrix**

$$\left[\widehat{A}_{ff}\ ,\ \widehat{A}_{fc}\right] = \left[A_{ff}\,, A_{fc}\,, A_{fX}\right] \begin{bmatrix} I & 0 \\ 0 & I \\ E_{Xf} & E_{Xc} \end{bmatrix}$$

that is,

$$\widehat{A}_{ff} = A_{ff} + A_{fX} E_{Xf}$$

and

$$\widehat{A}_{fc} = A_{fc} + A_{fX} E_{Xc}$$

Prolongation in Element-free AMGe: based on extensions

- **Then the *i*th row of the prolongation matrix *P* is taken as the *i*th row of the matrix**

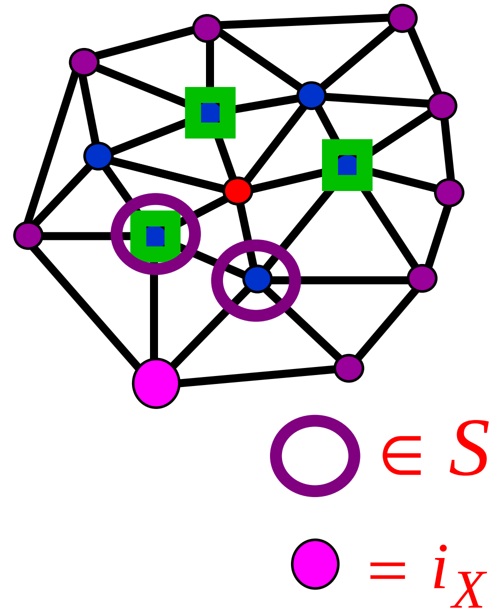
$$-\left(\widehat{A}_{ff}^{-1} \widehat{A}_{fc} \right)$$

- **To make use of this method we must determine useful ways in which to build the extension operator**

$$E = \begin{bmatrix} I & 0 \\ 0 & I \\ E_{Xf} & E_{Xc} \end{bmatrix}$$

Examples of extension: *A*-extension

- Given v defined on $\Omega(i)$, we wish to extend it to v_X defined on $\Omega_X(i)$.



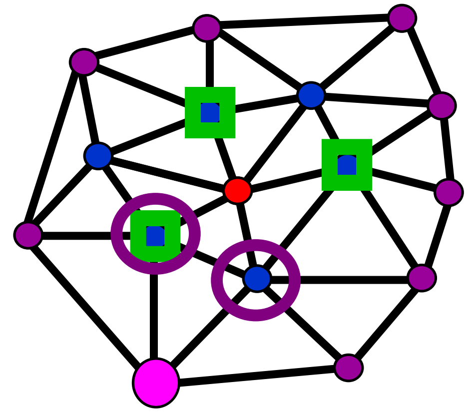
- Let i_X be an exterior dof and define $S = \{j : a_{i_X, j} \neq 0\}$ to be entries of *A* to which i_X is connected.

- .The *A*-extension is:

$$v_{i_X} = \frac{1}{\sum_S |a_{i_X,j}|} \sum_S |a_{i_X,j}| v_j$$

Examples of extension: L_2-extension

- Given v defined on $\Omega(i)$, we wish to extend it to v_X defined on $\Omega_X(i)$.



- Let i_X be an exterior dof and define $S = \{j : a_{i_X, j} \neq 0\}$ to be entries of ***A*** to which i_X is connected.

$\bigcirc \in S$

$\bullet = i_X$

- The L_2-extension is (a simple average):

$$v_{i_X} = \frac{1}{\sum_S 1} \sum_S v_j$$

Examples of extensions: based on minimizing a quadratic functional

- **This is the method Achi Brandt proposed in 1999.**
- **Given v defined on $\Omega(i)$, find the extension v_X by finding**

$$min\ Q(v_X)$$

where $$Q(v_X) = \sum_{\substack{i_X \in \Omega_X(i) \\ j \in \Omega(i)}} |a_{i_X, j}| (v_{i_X} - v_j)^2$$

- **This is a "simultaneous" extension, and is more expensive than A-extension or L_2-extension**

Examples of extensions: minimizing a "cut-off" quadratic functional

- Given v defined on $\Omega(i)$, find the extension v_X that satisfies

$$\min_{v_X} [Bv]^T A_{\overline{\Omega}(i)} [Bv]$$

where $v = \begin{pmatrix} v_f \\ v_c \\ v_X \end{pmatrix}$ and $B = \begin{bmatrix} I & & \\ & I & \\ & & \theta_X \end{bmatrix}$ is a diagonal matrix. A good choice is the vector

$$\theta_X = -A_{XX}^{-1} \left[A_{Xf} \, , \; A_{Xc} \right] \begin{bmatrix} 1_f \\ 1_c \end{bmatrix}$$

note that this is also a "simultaneous" extension (but less expensive than the "full" quadratic)

Classical AMG viewed as extension

- **The classical (Ruge-Stüben) AMG corresponds to selecting**

$$\Omega(i) = \{i\} \cup \Omega_C(i)$$

and defining an A-extension by setting $v_{i_X} = v_i$ if i_X is weakly connected to i, and setting

$$v_{i_X} = \frac{1}{\sum a_{i_X,j}} \sum a_{i_X,j} v_j$$

if i_X is strongly connected to i

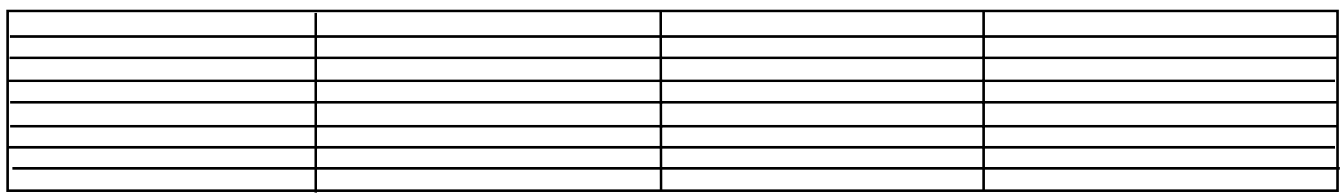
Properties of the extensions

- All the extensions described ensure that if v is constant in $\Omega(i)$ then the extension v_X is the same constant in $\Omega_X(i)$, an important property for second-order elliptic PDEs.

- For systems of PDEs we use the same extension mappings, based on the blocks of A associated with a given physical variable. That is, the extension to a dof i_X corresponding to the physical variable k is based on dofs from $\Omega(i)$ that describe the same physical variable k.

A simple example: the stretched quadrilateral

- **Consider the 2-D Laplacian operator, finite-element formulation on regular quadrialteral elements that are greatly stretched:** $h_x \gg h_y$

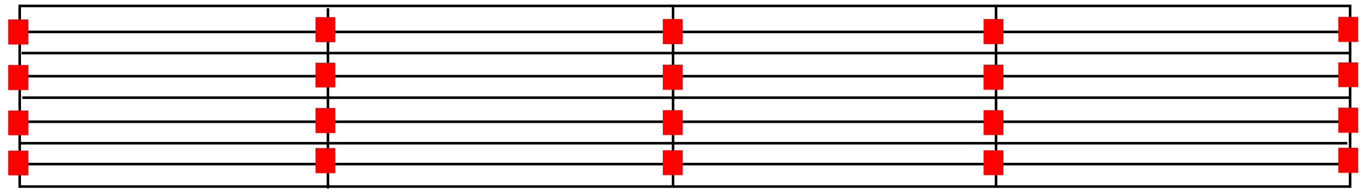


- **As** $\dfrac{h_x}{h_y} \rightarrow \infty$ **, the operator stencil tends to:**

$$\begin{bmatrix} -1 & -4 & -1 \\ 2 & 8 & 2 \\ -1 & -4 & -1 \end{bmatrix}$$

A simple example: the stretched quadrilateral: geometric approach

- **For this problem the standard geometric multigrid approach is to semicoarsen:**



- **And the interpolation stencil is:**

$$P_A = \begin{bmatrix} 0 & 0.5 & 0 \\ & * & \\ 0 & 0.5 & 0 \end{bmatrix}$$

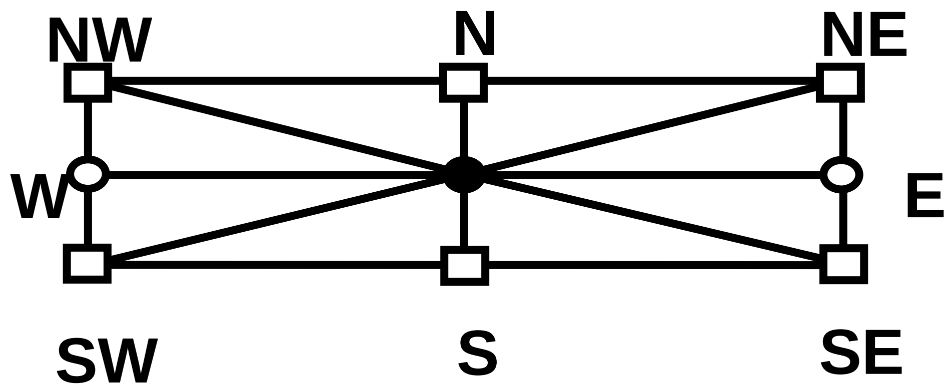
A simple example: the stretched quadrilateral: the AMG stencil

- **A simple calculation shows that classical AMG yields the interpolation stencil:**

$$P_{AMG} = \begin{bmatrix} \frac{1}{12} & \frac{1}{3} & \frac{1}{12} \\ & * & \\ \frac{1}{12} & \frac{1}{3} & \frac{1}{12} \end{bmatrix} \approx \begin{bmatrix} .083 & .333 & .083 \\ & * & \\ .083 & .333 & .083 \end{bmatrix}$$

A simple example: the stretched quadrilateral: the neighborhood

- **Let** $\Omega(i) = i \cup \Omega_C(i) = i \cup \{N, S, SW, NW, SE, NE\}$



- **Define** $\Omega_\chi(i) = \{ W, E \}$

then $A_{ff} = [\,8\,]$

$A_{fc} = [\,-4 \;\; -4 \;\; -1 \;\; -1 \;\; -1 \;\; -1\,]$

$A_{fX} = [\,2 \;\; 2\,]$

A simple example: the stretched quadrilateral: the *A-extension*

- **For the A-extension the extension operators are:**

$$E_{Xc} = \frac{1}{12}\begin{bmatrix} 1 & 1 & 4 & 4 & & \\ 1 & 1 & & & 4 & 4 \end{bmatrix} \qquad E_{Xf} = \frac{1}{12}\begin{bmatrix} 2 \\ 2 \end{bmatrix}$$

from which $\widehat{A}_{ff} = \begin{bmatrix} \frac{104}{12} \end{bmatrix}$ **, and**

$$\widehat{A}_{fc} = \frac{1}{3}\begin{bmatrix} -11 & -11 & -1 & -1 & -1 & -1 \end{bmatrix}$$

yielding the interpolation stencil:

$$P_A = \begin{bmatrix} \frac{1}{26} & \frac{11}{26} & \frac{1}{26} \\ & * & \\ \frac{1}{26} & \frac{11}{26} & \frac{1}{26} \end{bmatrix} \approx \begin{bmatrix} .038 & .423 & .038 \\ & * & \\ .038 & .423 & .038 \end{bmatrix}$$

A simple example: the stretched quadrilateral: the L_2-*extension*

- A similar calculation for the weights using the L_2 extension yields the interpolation stencil:

$$P_{L_2} = \begin{bmatrix} \frac{3}{44} & \frac{16}{44} & \frac{3}{44} \\ & * & \\ \frac{3}{44} & \frac{16}{44} & \frac{3}{44} \end{bmatrix} \approx \begin{bmatrix} .068 & .364 & .068 \\ & * & \\ .068 & .364 & .068 \end{bmatrix}$$

A simple example: the stretched quadrilateral: the *cut-off quadratic min*

- **For the cut-off extension the extension operators are:**

$$E_{Xc} = \frac{1}{8}\begin{bmatrix} 1 & 1 & 4 & 4 & & \\ 1 & 1 & & & 4 & 4 \end{bmatrix} \qquad E_{Xf} = -\frac{1}{4}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

from which $\widehat{A}_{ff} = \begin{bmatrix} 7 \end{bmatrix}$ **, and**

$$\widehat{A}_{fc} = \frac{1}{2}\begin{bmatrix} -7 & -7 & 0 & 0 & 0 & 0 \end{bmatrix}$$

yielding the interpolation stencil:

$$P_A = \begin{bmatrix} 0 & .5 & 0 \\ & * & \\ 0 & .5 & 0 \end{bmatrix}$$

The stencil produced by various extension methods:

- **Classical AMG** ⇒ $$P_{AMG} = \begin{bmatrix} .083 & .333 & .083 \\ & * & \\ .083 & .333 & .083 \end{bmatrix}$$

- $$P_A = \begin{bmatrix} .038 & .423 & .038 \\ & * & \\ .038 & .423 & .038 \end{bmatrix}$$ ⇐ **A-extension**

- **L2-extension** ⇒ $$P_{L_2} = \begin{bmatrix} .068 & .364 & .068 \\ & * & \\ .068 & .364 & .068 \end{bmatrix}$$

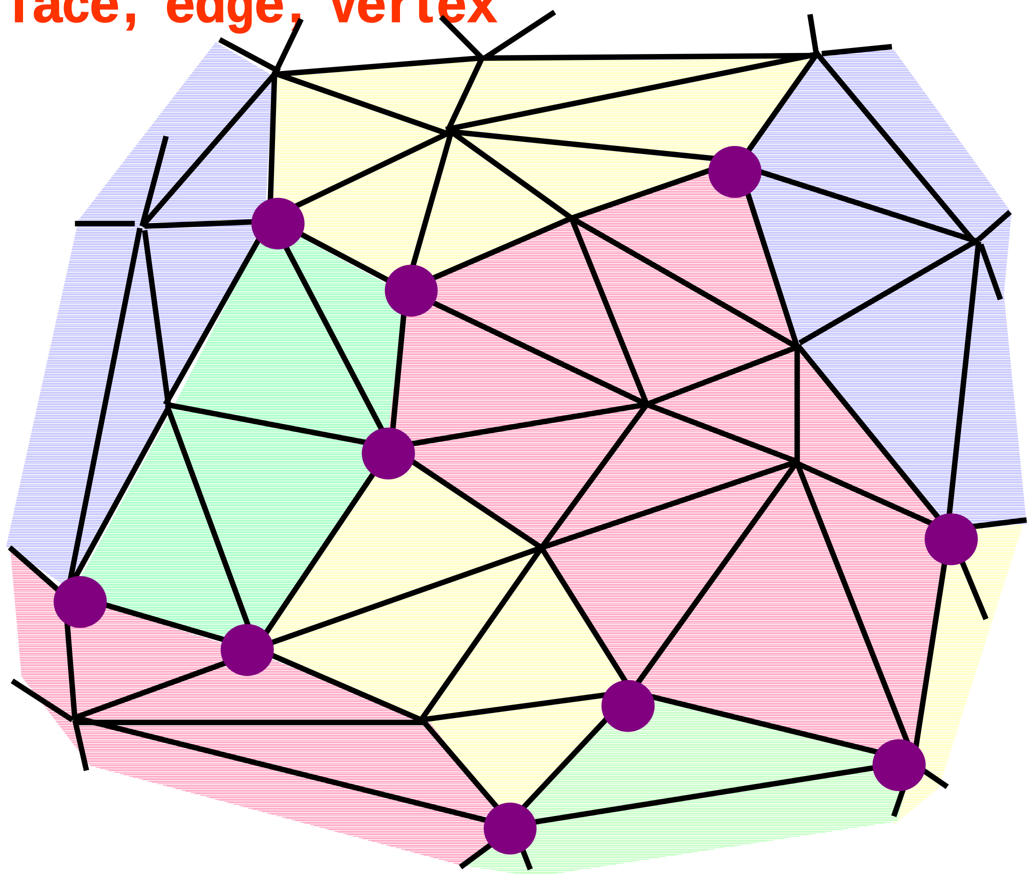
- $$P_A = \begin{bmatrix} 0 & .5 & 0 \\ & * & \\ 0 & .5 & 0 \end{bmatrix}$$ ⇐ **cut-off quadratic**

Numerical experiments

- **Second order elliptic operator**
  - **Unstructured triangular mesh (400, 1600, 6400, and 25600 fine-grid elements)**
  - **coarsened using agglomeration method of Jones & Vassilevski**

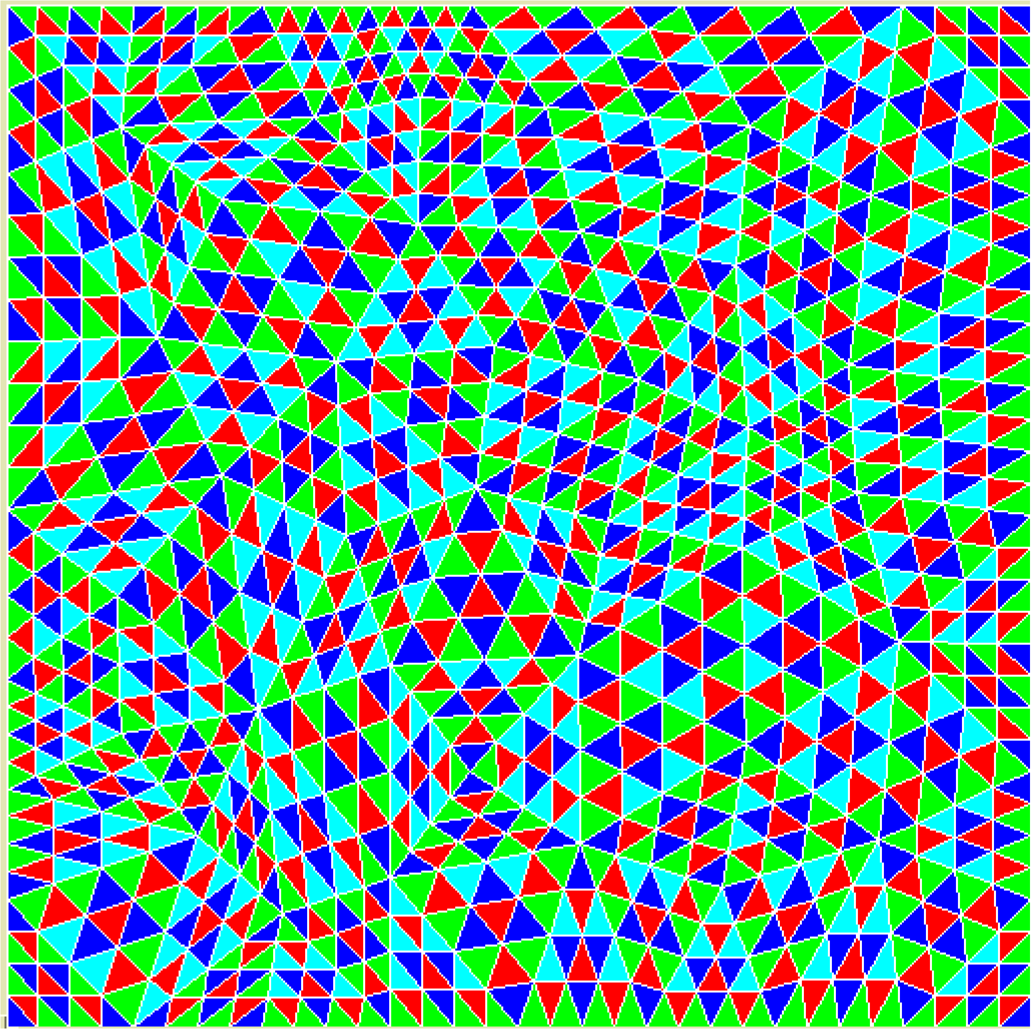
AMGe requires coarse-grid elements & stiffness matrices.

- **Element Agglomeration:** **Use graph theory to create coarse elements first, then select coarse-grid by abstracting geometric concepts of face, edge, vertex**



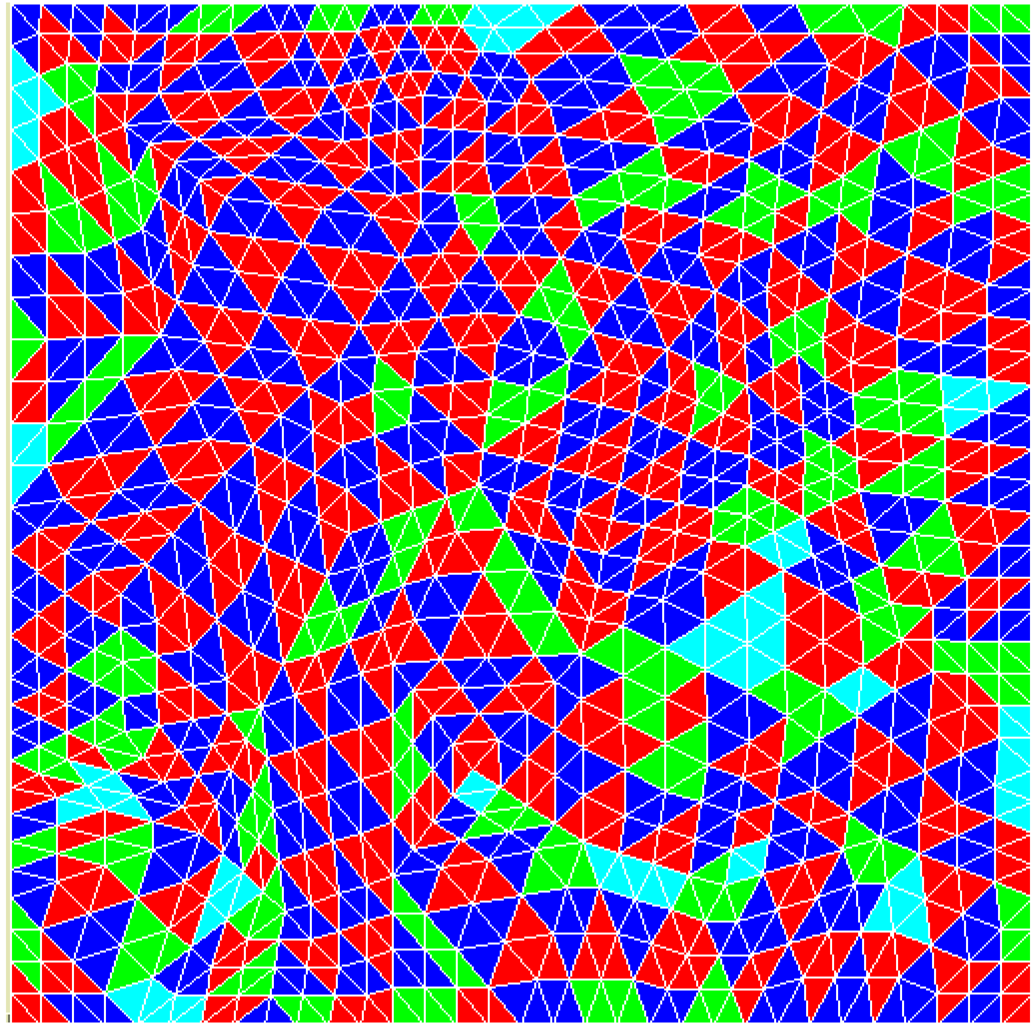
Agglomeration coarsening

- **Finest grid**
  - **1600 elements**
  - **861 dofs**
  - **5781 nonzero entries**



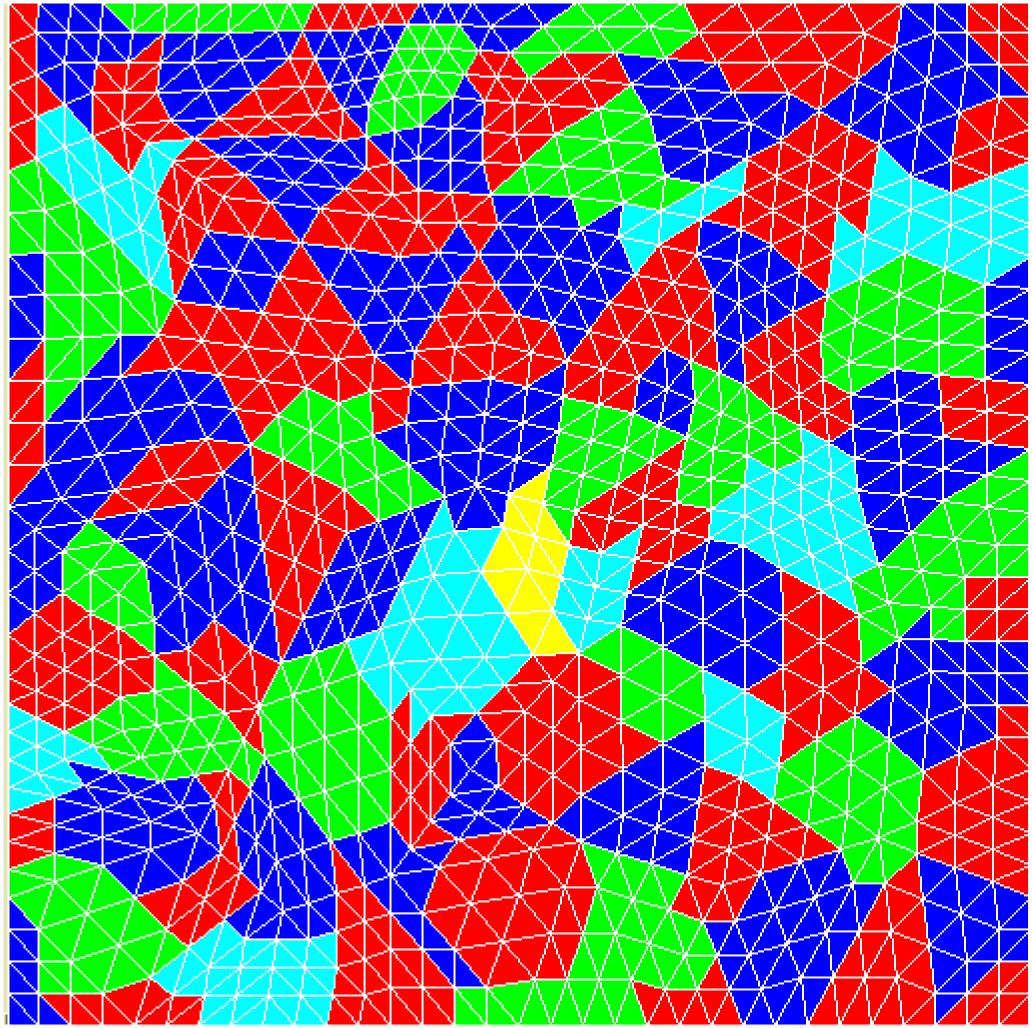
Agglomeration coarsening

- 1st coarse grid
  - 382 elements
  - 330 dofs
  - 2602 nonzero entries



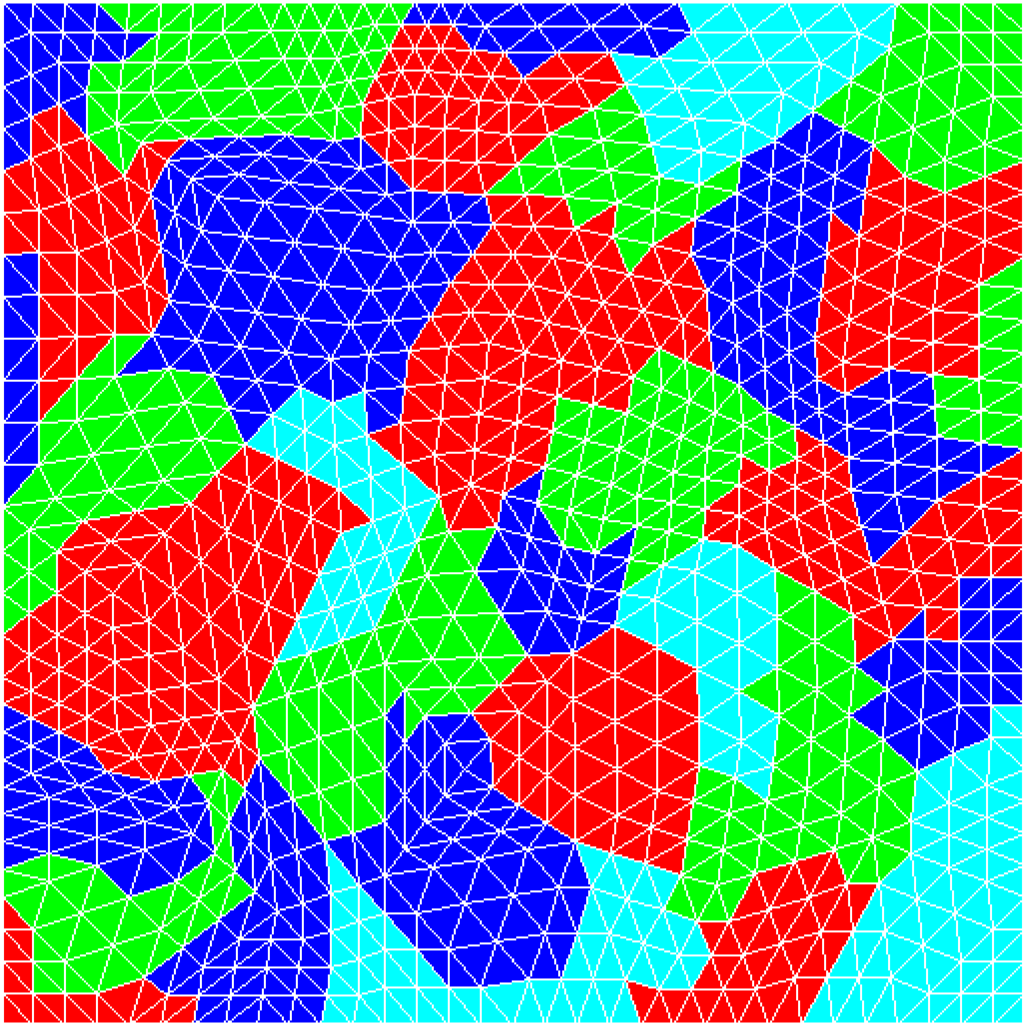
Agglomeration coarsening

- **2nd coarse grid**
  - **93 elements**
  - **143 dofs**
  - **1397 nonzero entries**



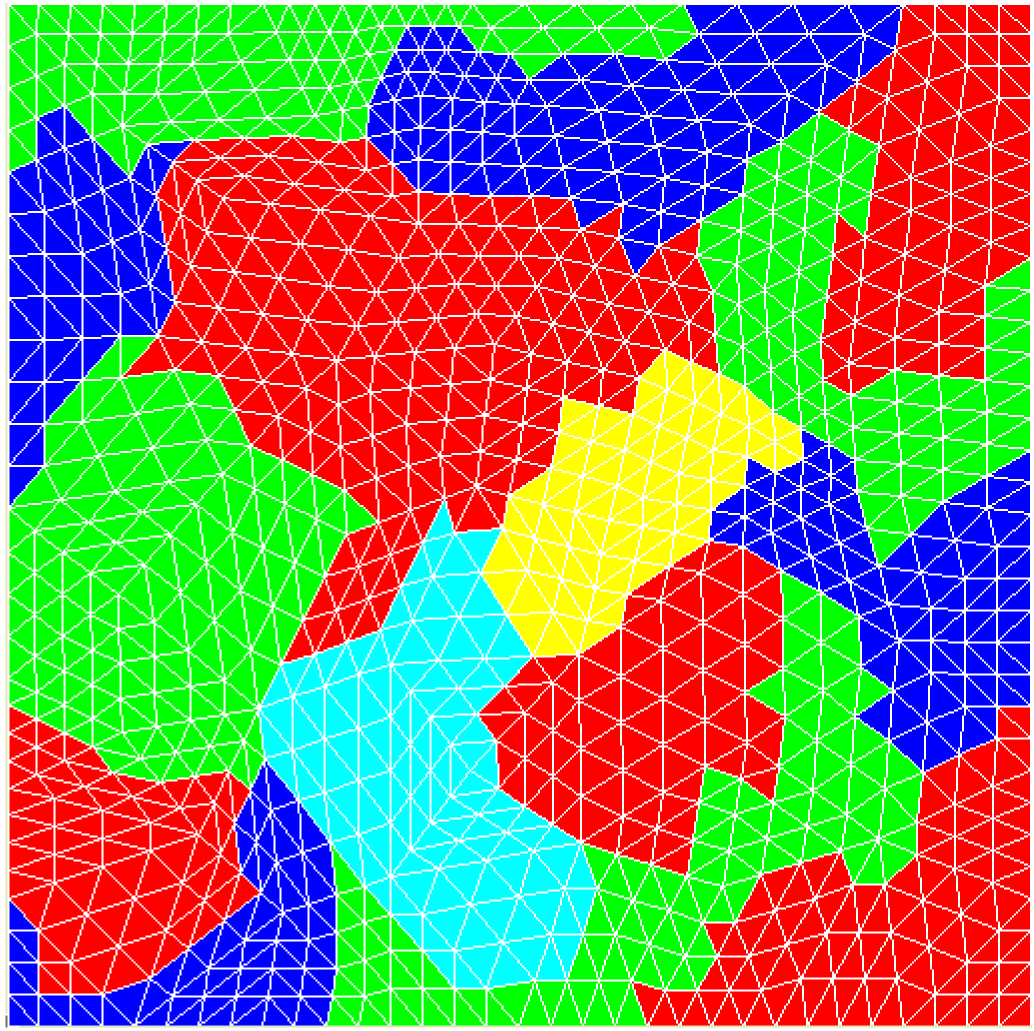
Agglomeration coarsening

- **3rd coarse grid**
  - **33 elements**
  - **64 dofs**
  - **634 nonzero entries**



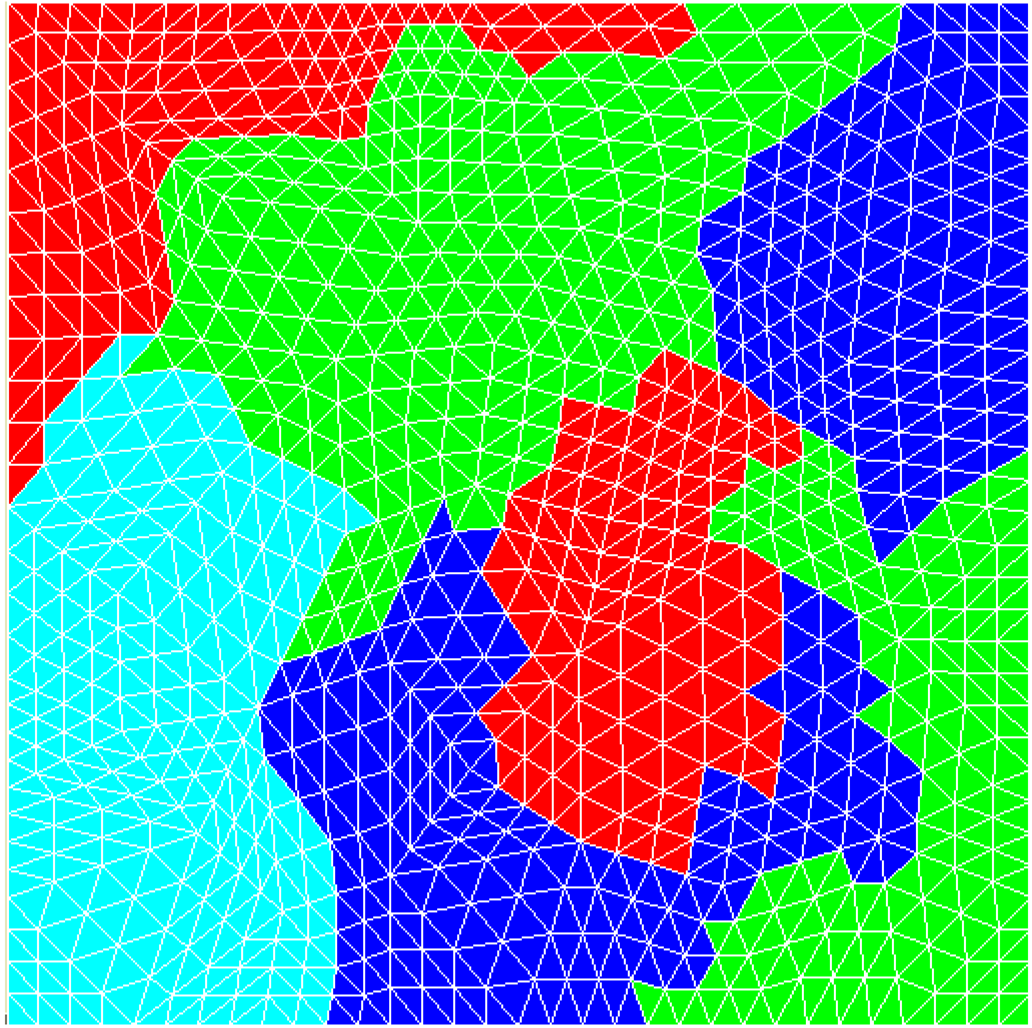
Agglomeration coarsening

- 4th coarse grid
  - 15 elements
  - 32 dofs
  - 304 nonzero entries



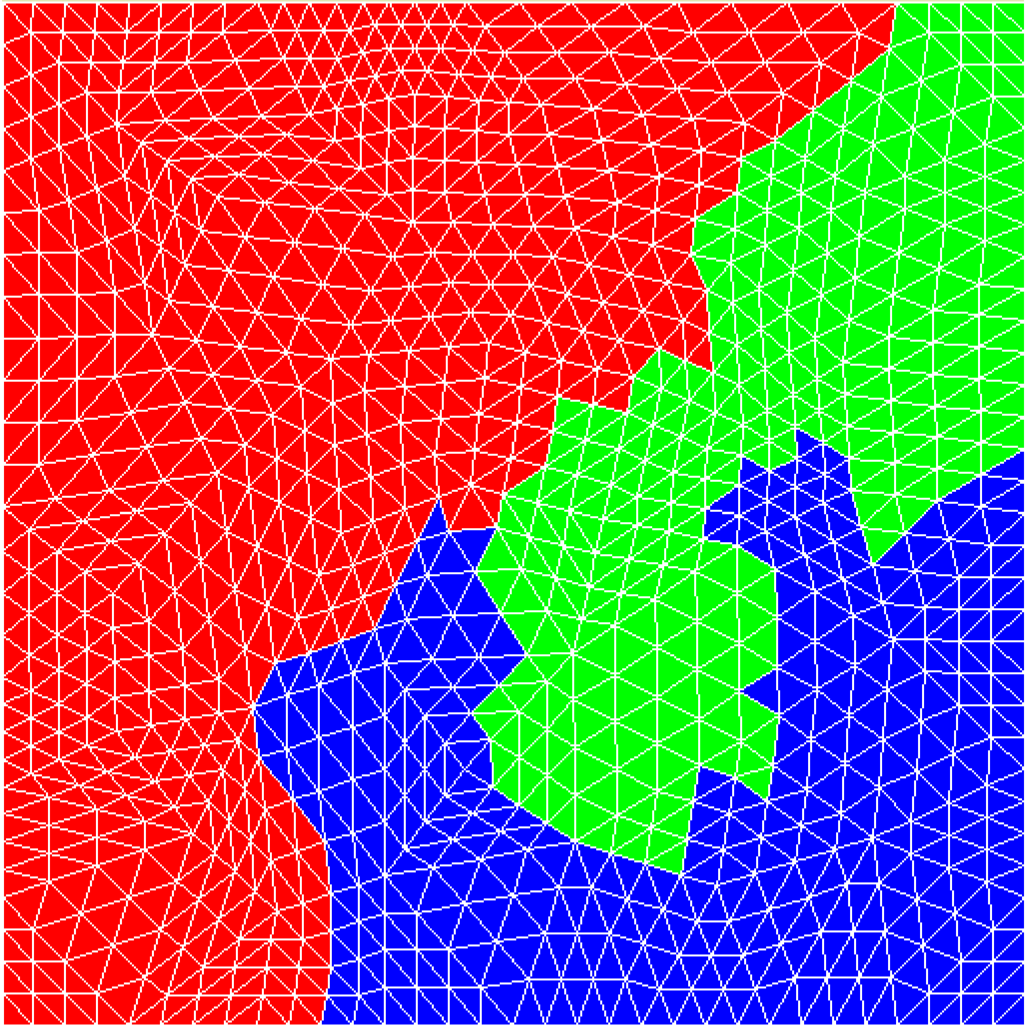
Agglomeration coarsening

- 5th coarse grid
  - 7 elements
  - 16 dofs
  - 126 nonzero entries



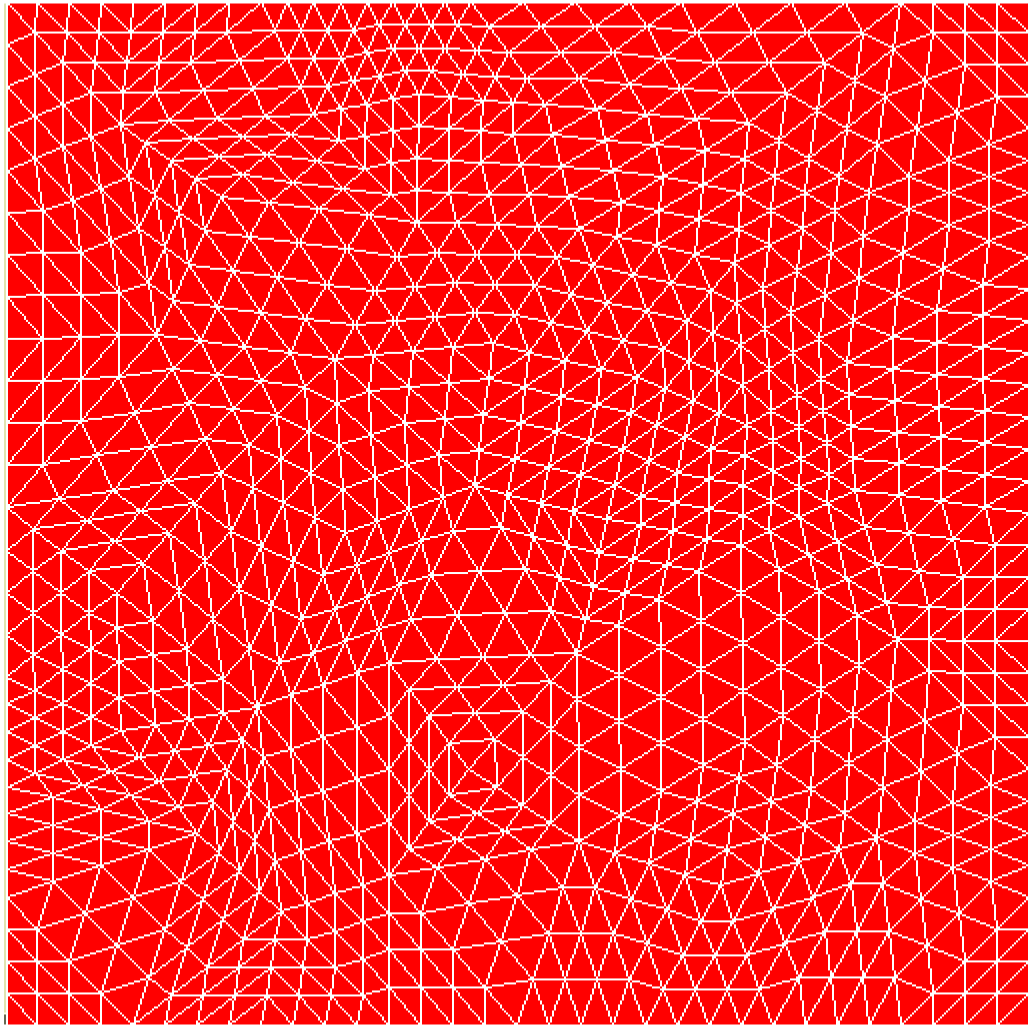
Agglomeration coarsening

- **6th coarse grid**
  - **3 elements**
  - **8 dofs**
  - **46 nonzero entries**



Agglomeration coarsening

- **coarsest grid**
  - **1 elements**
  - **4 dofs**
  - **16 nonzero entries**



Coarsening history, unstructured 2nd order PDE

- **unstructured triangular grid**
- **2nd order PDE**

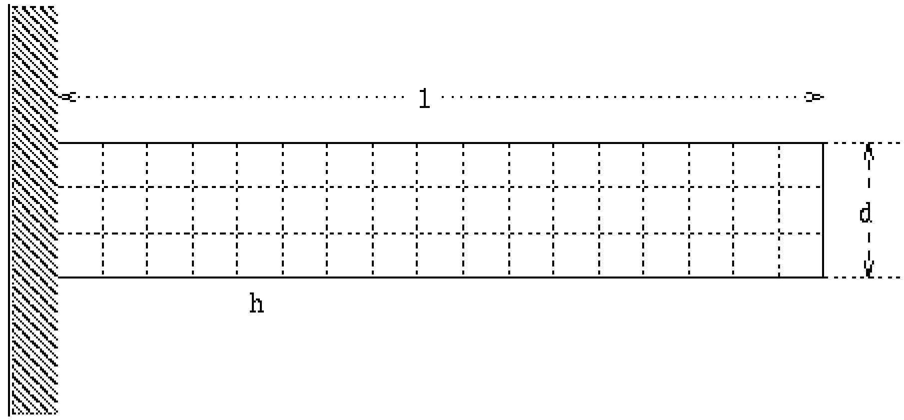
level #	grid	# 4	# 3	# 2	# 1
0	matrix size	90321	22761	5781	1491
	# dofs	13041	3321	861	231
	# elements	25600	6400	1600	400
1	matrix size	32898	9540	2602	1094
	# dofs	4108	1152	330	114
	# elements	6013	1427	382	76
2	matrix size	14305	4361	1397	470
	# dofs	1507	451	143	50
	# elements	1489	374	93	26
3	matrix size	7193	2098	634	199
	# dofs	643	198	64	23
	# elements	392	117	33	11
4	matrix size	3458	975	304	88
	# dofs	302	91	32	12
	# elements	158	47	15	5
5	matrix size	1580	453	126	36
	# dofs	140	45	16	6
	# elements	70	22	7	2
6	matrix size	714	188	46	16
	# dofs	68	22	8	4
	# elements	33	10	3	1
7	matrix size	274	84	16	
	# dofs	30	12	4	
	# elements	14	5	1	
8	matrix size	120	30		
	# dofs	16	6		
	# elements	7	2		
9	matrix size	42	16		
	# dofs	8	4		
	# elements	3	1		
10	matrix size	16			
	# dofs	4			
	# elements	1			

Convergence results, unstructured 2nd order PDE

- **2nd order PDE problem**
  - **unstructured triangular grid**
  - **V(1,1) cycles**
  - **Gauß-Seidel smoothing**

Interp. rule	# elements	= 400	= 1600	= 6400	= 25600
nonconf. AMGe	# iter ϱ	14 0.115	16 0.172	21 0.252	23 0.289
A-extension	# iter ϱ	13 0.118	15 0.158	19 0.218	20 0.247
L_2-extension	# iter ϱ	13 0.119	16 0.161	19 0.227	21 0.249
quadr. funct. min.	# iter ϱ	13 0.105	15 0.152	19 0.222	19 0.231

Numerical experiments



- 2d elasticity (including thin-body)
  - structured quadrilateral grid, $h_x = h_y$
  - $d \in (0, 1]$ is the beam thickness
  - coarsened using agglomeration method
  - the same coarse-grid is used for AMGe (vertices of agglomerated elements)

Coarsening history, elasticity problem

- **Structured rectangular grid**
- **2-d elasticity**
- **d = 1**

level #		$h = 0.050$	$h = 0.025$	$h = 0.0125$
0	size	14884	58564	232324
	# dofs	882	3362	13122
1	size	10440	40880	161760
	# dofs	264	924	3444
2	size	4128	17248	70488
	# dofs	84	264	924
3	size	1000	4956	19056
	# dofs	32	94	284
4	size	256	1404	6128
	# dofs	16	38	104
5	size	64	324	1668
	# dofs	8	18	42
6	size		144	576
	# dofs		12	24
7	size		64	144
	# dofs		8	12
8	size			64
	# dofs			8

Coarsening history, elasticity problem

- **Structured rectangular grid**
- **2-d elasticity**
- **thin body**
  - **d = 0.05**

level #		$h = 0.025$	$h = 0.0125$	$h = 0.00625$
0	size	3388	12532	48100
	# dofs	246	810	2898
1	size	1664	7328	30656
	# dofs	88	252	820
2	size	784	3744	10152
	# dofs	44	132	252
3	size	384	1152	3816
	# dofs	24	48	132
4	size	144	384	1152
	# dofs	12	24	48
5	size	64	144	384
	# dofs	8	12	24
6	size		64	144
	# dofs		8	12
7	size			64
	# dofs			8

Convergence results, elasticity

- **2d elasticity problem**
  - **d = 1**
  - **V(1,1) cycles**
  - **Gauß-Seidel smoothing**

Interp. rule		$h = 0.050$	$h = 0.025$	$h = 0.0125$
nonconf. AMGe	# iter	16	18	20
	ϱ	0.172	0.206	0.234
A-extension	# iter	12	12	12
	ϱ	0.099	0.098	0.097
L_2-extension	# iter	13	13	13
	ϱ	0.101	0.102	0.104

Convergence results, elasticity

- 2d thin-body elasticity problem
  - d = 0.05
  - V(1,1) cycles
  - Gauß-Seidel smoothing

Interp. rule		$h = 0.025$	$h = 0.0125$	$h = 0.0062$
nonconf. AMGe	# iter	17	18	19
	ϱ	0.180	0.198	0.22
A-extension	# iter	20	23	22
	ϱ	0.227	0.286	0.280
L_2-extension	# iter	18	20	27
	ϱ	0.203	0.243	0.254

Conclusions

- AMGe produces superior prolongation, but requires extra information, i.e., the element matrices
- For purely algebraic problems, element-free AMGe provides a technique for building pseudo-element matrices based on several extension schemes
- Preliminary experimental results suggest that element-free AMGe also produces superior prolongation, competitive to AMGe results
- Systems of PDEs can be handled in a very natural way

Some papers of interest

- Pre/Re-prints Available at
  http://www.llnl.gov/CASC/linear_solvers

- **V. Henson** and **P. Vassilevski**, "**Element-free AMGe: General Algorithms for computing Interpolation Weights in AMG**," submitted to *SIAM Journal on Scientific Computing*.
- M. Brezina, A. Cleary, R. Falgout, **V. Henson**, J. Jones, T. Manteuffel, S. McCormick, and J. Ruge, "**Algebraic Multigrid Based on Element Interpolation (AMGe)**," to appear in *SIAM Journal on Scientific Computing*.
- J. Jones and **P. Vassilevski**, "**AMGe Based on Element Agglomeration,**" submitted to *SIAM Journal on Scientific Computing*.
- A. Cleary, R. Falgout, **V. Henson**, J. Jones, T. Manteuffel, S. McCormick, G. Miranda, and J. Ruge, "**Robustness and Scalability of Algebraic Multigrid**," *SIAM Journal on Scientific Computing,* vol. 21 no. 5, pp. 1886-1908, 2000
- A. Cleary, R. Falgout, **V. Henson**, J. Jones, "**Coarse-Grid Selection for Parallel Algebraic Multigrid**," in *Proc. of the Fifth International Symposium on: Solving Irregularly Structured Problems in Parallel*, Lecture Notes in Computer Science, Springer-Verlag, New York, 1998.