Some Studies on Algebraic Multigrid (AMG)

Klaus Stüben

International Workshop on
„Algebraic Multigrid Methods“,
St. Wolfgang, June 26-28, 2000

Overview

- “Classical” AMG
- Mature cases, performance
- Critical cases, discussion and remedies
  - Large positive couplings (bilinear FE)
  - Small eigenvalues (linear elasticity)
- Results

Algebraic Multigrid (AMG)

"Classical" AMG

Mimics geometric multigrid to solve sparse, linear equations (here s.p.d.)

$$A_h u^h = f^h \qquad \sum_j a_{ij}^h u_j^h = f_i^h \quad (i \in \Omega_h)$$

without exploiting geometric information

Components of AMG

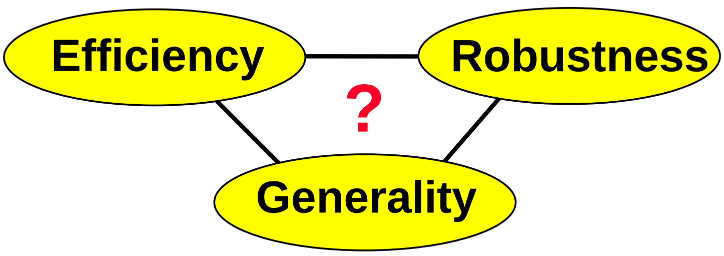
- Smoothing by variable-wise GS relaxation
- Coarsening based on strong connectivity
- Interpolation based on matrix-coefficients
  (Restriction = transpose of interpolation)
- Galerkin coarse-level operators

Hierarchical Approaches

Efficient solution requires hierarchical approaches!

However:

How far can we get without exploiting geometry?



Tradeoff between

Computational cost and memory
(speed of coarsening, sparsity on coarse levels)

Convergence and robustness
(quality of interpolation)

Classical Applications for AMG

M-matrices: $\sum_j a_{ij}^h \geq 0, \quad a_{ij}^h \leq 0 \ (i \neq j), \quad a_{ii}^h > 0$

Local property of error after smoothing

$$\sum_j a_{ij}^h e_j^h \approx 0 \quad (i \in \Omega_h) \Longrightarrow e_{ii}^h \approx \frac{1}{a_{ii}^h} \sum_{j \neq i} |a_{ij}^h| e_j^h \quad (i \in \Omega_h)$$

Local average!

Error is smooth in the direction of large (negative!) couplings: "strong couplings"

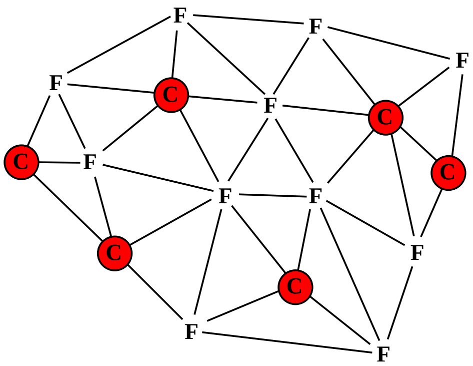
Coarsening

Strong couplings

i is "strongly coupled" to j if $a_{ij} < 0$ and

$$-a_{ij} \geq \varepsilon_{str} \max\{ |a_{ik}| : a_{ik} < 0 \}$$

Coarsening "in the direction" of strong couplings



graph of strong couplings

C/F-splitting:

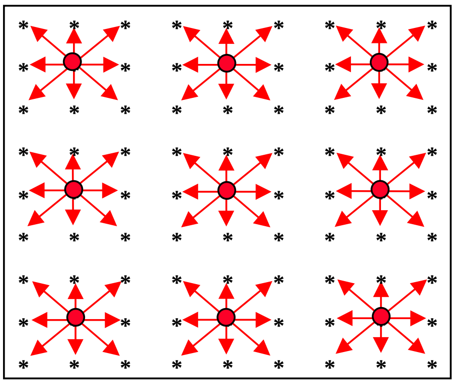
$\Omega_h = F_h \cup C_h$ fine level $\Longrightarrow$ $\Omega_H = C_h$ coarse level

C_h: maximally independent set of variables (w.r.t. graph defined by strong couplings)

Coarsening

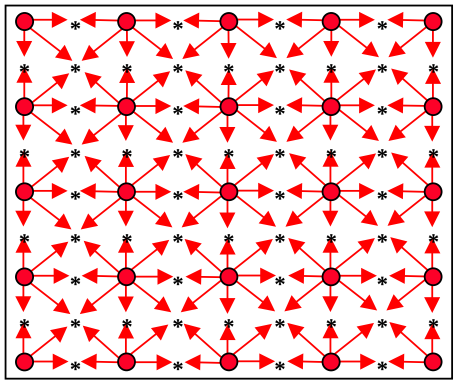
$$A^h \mathrel{\hat{=}} \frac{1}{3h^2}\begin{bmatrix} -1 & -1 & -1 \\ -1 & 8 & -1 \\ -1 & -1 & -1 \end{bmatrix}_h$$

minimal # of C-variables



aggregation based AMG

maximal # of C-variables

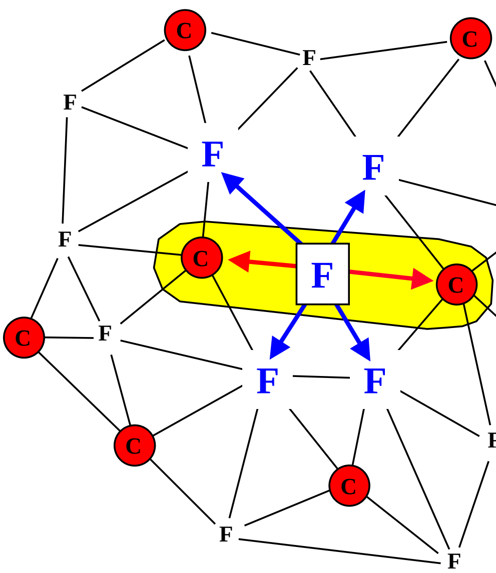


classical AMG

Interpolation

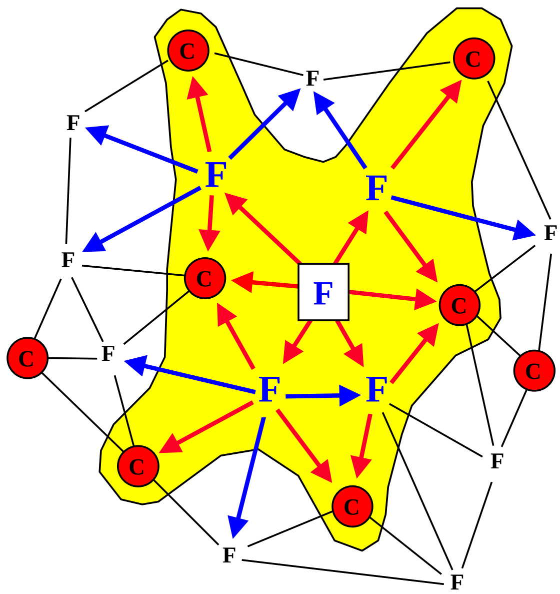
Direct interpolation:

Interpolate from direct C-neighbors only



Standard interpolation:

Eliminate neighboring F-couplings



Afterwards: truncation of “small” interpolation weights!!

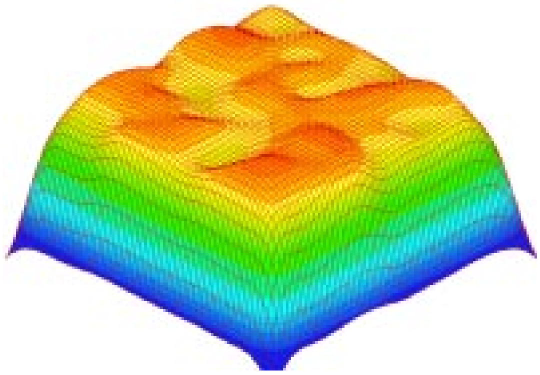
Classical Applications for AMG

Examples

$$\begin{bmatrix} & -1 & \\ -1 & 4 & -1 \\ & -1 & \end{bmatrix} e_0^h \approx 0$$

$$e_0^h \approx (e_N^h + e_S^h + e_W^h + e_E^h)/4$$

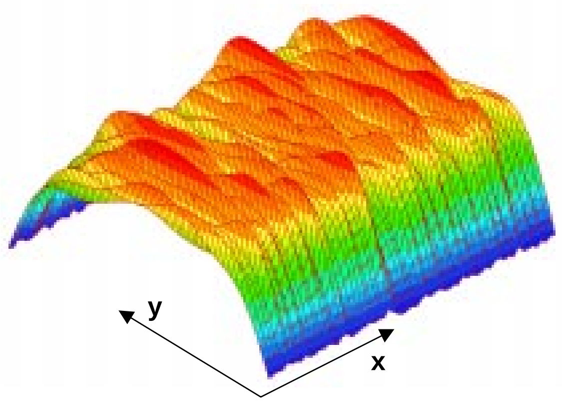
strong couplings



$$\begin{bmatrix} & -1 & \\ -\varepsilon & 2(1+\varepsilon) & -\varepsilon \\ & -1 & \end{bmatrix} e_0^h \approx 0$$

$$e_0^h \approx (e_N^h + e_S^h + \varepsilon e_W^h + \varepsilon e_E^h)/2(1+\varepsilon)$$

strong couplings

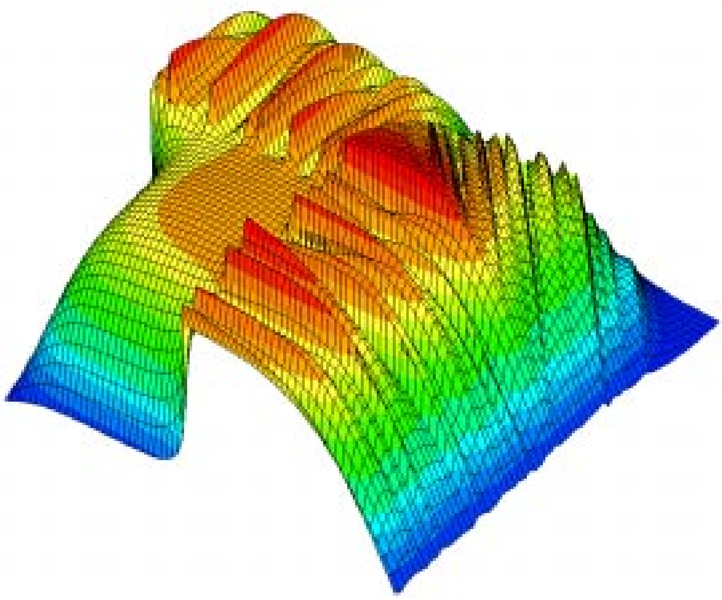


Classical Applications for AMG

$$-(au_x)_x - (bu_y)_y + cu_{xy} = f$$

$a=1$, $b=10^3$, $c=0$	$a=1$, $b=1$, $c=2$
$a=1$, $b=1$, $c=0$	$a=10^3$, $b=1$, $c=0$

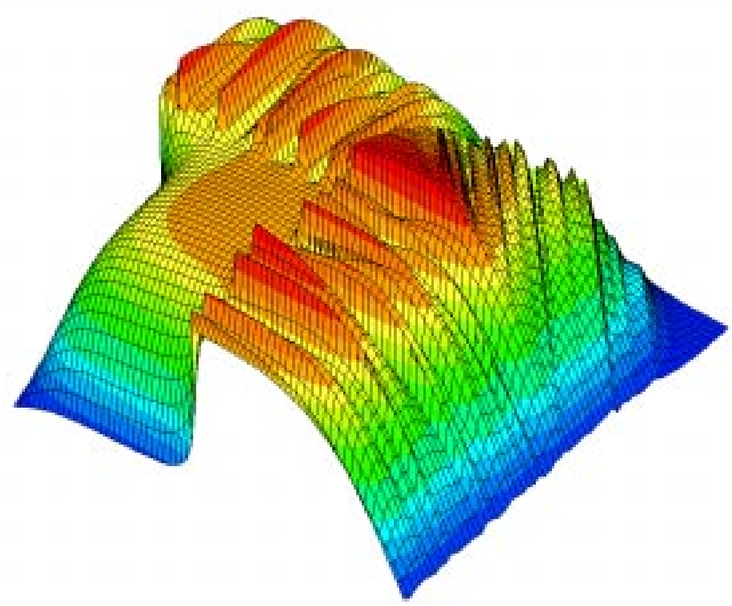
Discontinuous coefficients,
strong anisotropies

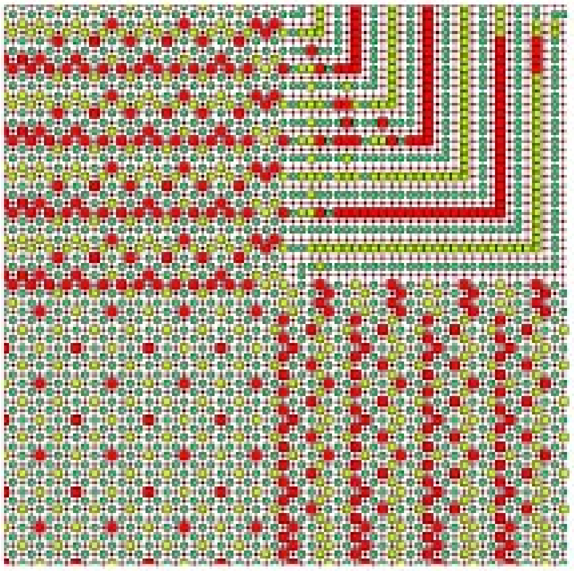


"Smooth" error (pointwise relaxation)

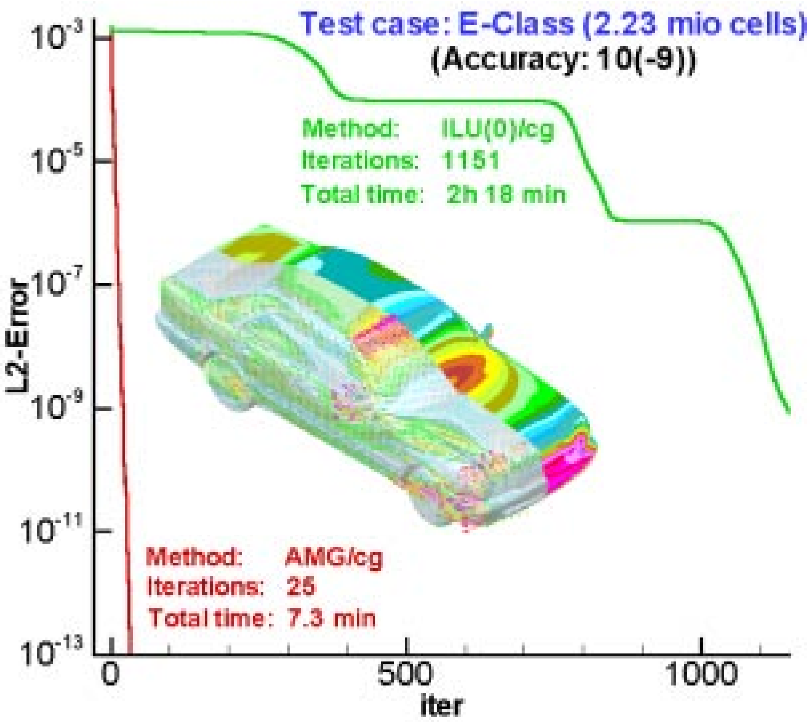
Classical Applications for AMG

Locally adapted AMG coarsening,
operator dependent interpolation

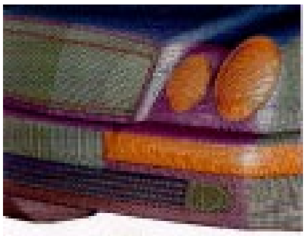




Performance of AMG



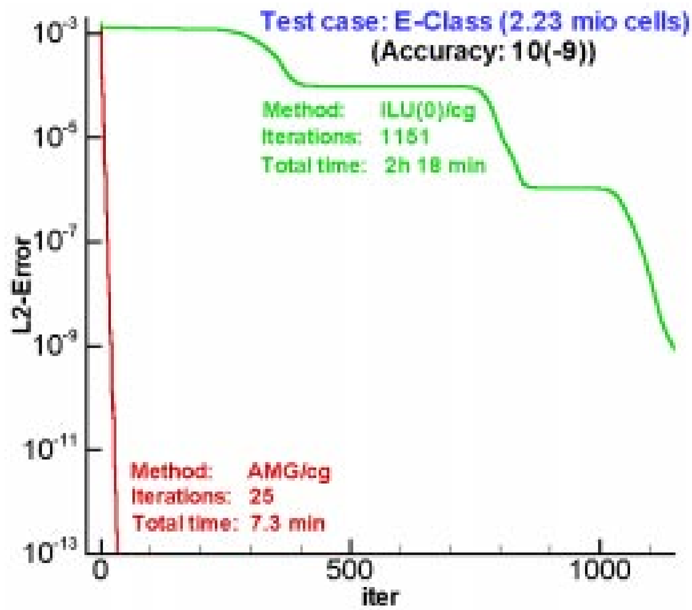
$$\text{A-complexity} = \sum_i |A_i| \ / \ |A_1| \ = 1.46$$

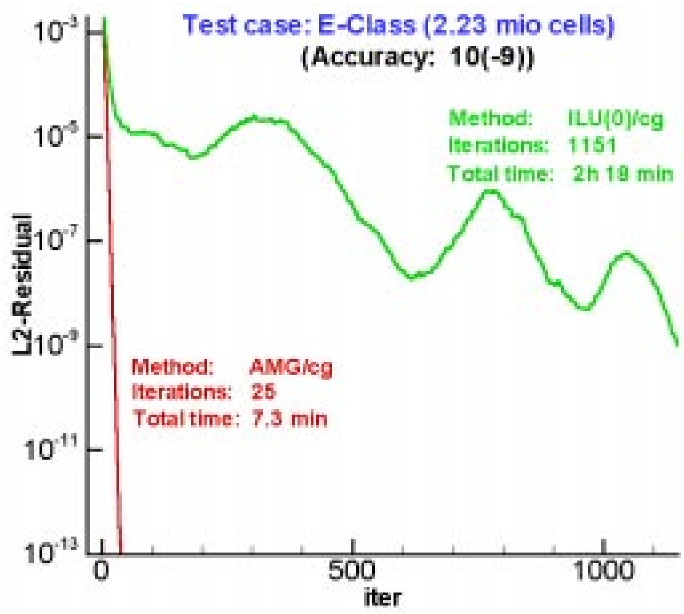


E-Class (2,23 mio cells)
Mercedes-Benz, Computational Dynamics

Performance of AMG

Residual versus error reduction



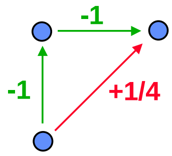


Non-M-Matrices

Near M-matrix problems

"Small" positive coefficients can be ignored

$$\begin{bmatrix} -\frac{1}{4} & -1 & +\frac{1}{4} \\ -1 & 4 & -1 \\ +\frac{1}{4} & -1 & -\frac{1}{4} \end{bmatrix}$$



Weakly diagonally dominant matrices	interpolation weights:
Large negative couplings $\rightarrow$ smoothness	positive
Large positive couplings $\rightarrow$ „strict" oscillations	negative

$$\begin{bmatrix} & -1 & \\ +1 & 4 & +1 \\ & -1 & \end{bmatrix} \qquad \begin{bmatrix} & +1 & \\ +1 & 4 & +1 \\ & +1 & \end{bmatrix}$$

More General Matrices

Sources of Convergence Problems

- Smoothing
  - There exists no well-defined direction of smoothness
  - AMG does not detect the direction of smoothness
- Coarse-level correction
  - accuracy of interpolation is insufficient for “relevant” error components

Investigation of model situations

Small Eigenvalues

AMG interpolation

$$e_C \rightarrow \begin{pmatrix} e_F \\ e_C \end{pmatrix}: \quad e_F^{(i)} = (I_{FC} e_C)^{(i)} = \sum_{j \in P_i} w_{ij} e_C^{(j)} \quad (i \in F)$$

Condition for h-independent two-level convergence

$$||e_F - I_{FC} e_C||_D^2 \le \tau ||e||_A^2 \quad \text{for all} \quad e = (e_F, e_C)^T$$

Application to eigenvectors of A

$$A\phi = \lambda\phi \quad (||\phi|| = 1)$$

$$||\phi_F - I_{FC}\phi_C||_D^2 \le \lambda\tau$$

The smaller λ, the higher the required „accuracy“!

Unless φ≈1, problems have to be expected!

Small Eigenvalues

Example:

$$A_c u \;\hat{=}\; -\Delta_h u - cu \quad (0 \le c < \lambda_{\min},\ \text{fixed } h)$$

$\lambda_{\min}$ smallest eigenvalue of A_0

$$A_0 \phi = \lambda_{\min} \phi \quad (||\phi|| = 1)$$

$$A_c \phi = (\lambda_{\min} - c)\phi$$

$$||\phi_F - I_{FC}\phi_C||_D^2 \le \tau(\lambda_{\min} - c) \;\rightarrow\; 0 \quad (c \rightarrow \lambda_{\min})$$

AMG for Systems of PDEs

Classical AMG:

- Point (or block) approach
  - Formally straightforward
- "Unknown" approach (separate treatment of physical unknowns)
  - very simple extension of scalar AMG

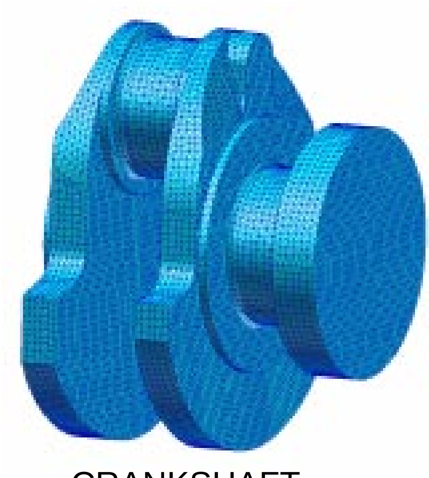
Closely related:

- Aggregation based AMG *(Vanek, Mandel)*
  - Testfunction-based interpolation
- AMGe *(Ruge & LLNL-group)*
  - Interpolation based on local stiffness matrices

In the following: unknown approach

Linear Elasticity

Computation of (small) displacements due to external forces



CRANKSHAFT
(MSC-AUDI)

Lamé equations

$$-2\mu\, div(\varepsilon(u)) - \lambda\, grad\, div(u) = f \quad (\Omega)$$

$$u = 0 \quad (\Gamma_0) \qquad \sigma(u)\bullet n = 0 \quad (\Gamma_1)$$

fixed boundary **free boundary**

$u=(u_1,u_2,u_3)$ displacements in $x=(x_1,x_2,x_3)$

$$\varepsilon = \begin{pmatrix} \varepsilon_{11} & \varepsilon_{12} & \varepsilon_{13} \\ \varepsilon_{21} & \varepsilon_{22} & \varepsilon_{23} \\ \varepsilon_{31} & \varepsilon_{32} & \varepsilon_{33} \end{pmatrix}$$

strain tensor: $\varepsilon_{ij} = \frac{1}{2}(\partial u_i / \partial x_j + \partial u_j / \partial x_i)$

$\sigma = C\varepsilon$ Hooke's law (σ = stress tensor)

ν = Poisson ratio $(0<\nu<1/2)$; steal: $\nu \approx 1/3$

Discretization: Bilinear finite elements
(Higher order: "p-solver" (Thole))

Linear Elasticity (2D)

Plane strain
(no strain in z-direction)

$$2\frac{1-\nu}{1-2\nu}u_{xx} + u_{yy} + \frac{1}{1-2\nu}v_{xy} = f_1$$
$$v_{xx} + 2\frac{1-\nu}{1-2\nu}v_{yy} + \frac{1}{1-2\nu}u_{xy} = f_2$$

$\nu=1/3$

Plane stress
(no stress in z-direction)

$$u_{xx} + \frac{1-\nu}{2}u_{yy} + \frac{1+\nu}{2}v_{xy} = f_1$$
$$\frac{1-\nu}{2}v_{xx} + v_{yy} + \frac{1+\nu}{2}u_{xy} = f_2$$

$\nu=1/2$

Linear Elasticity

Major problems:

- Anisotropies (large aspect ratios)
- Locking effects (bad discretization!)
- Nearly singular problems:
  The smaller the ratio of fixed and free boundary areas, the smaller the first eigenvalue of A

Eigenvalues in case of free boundaries:

Rigid body modes

$$\begin{pmatrix} u \\ v \\ w \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \qquad \begin{pmatrix} u \\ v \\ w \end{pmatrix} = \begin{pmatrix} 0 \\ z \\ -y \end{pmatrix}, \begin{pmatrix} z \\ 0 \\ -x \end{pmatrix}, \begin{pmatrix} y \\ -x \\ 0 \end{pmatrix}$$

Translations Rotations

9-point Poisson Discretization

Average of standard finite difference stencils:

$$u_{xx} = \frac{1}{1+2\alpha} \frac{1}{h_x^2} \begin{pmatrix} -1 & 2 & -1 \end{pmatrix}_{h_x} \begin{pmatrix} \alpha \\ 1 \\ \alpha \end{pmatrix}_{h_y} \sim \begin{pmatrix} -\alpha & 2\alpha & -\alpha \\ -1 & 2 & -1 \\ -\alpha & 2\alpha & -\alpha \end{pmatrix}$$

$$u_{yy} = \frac{1}{1+2\alpha} \frac{1}{h_y^2} \begin{pmatrix} \alpha & 1 & \alpha \end{pmatrix}_{h_x} \begin{pmatrix} -1 \\ 2 \\ -1 \end{pmatrix}_{h_y} \sim \begin{pmatrix} -\alpha & -1 & -\alpha \\ 2\alpha & 2 & 2\alpha \\ -\alpha & -1 & -\alpha \end{pmatrix}$$

positive definite: $-1/2 < \alpha \leq 1/2$

Standard finite differences: $\alpha = 0$
Bilinear finite elements: $\alpha = 1/4$

Isotropic 9-point Case

$$-u_{xx} - u_{yy} = f$$

$\alpha = -1/4$

$$\begin{pmatrix} 1 & -3 & 1 \\ -3 & 8 & -3 \\ 1 & -3 & 1 \end{pmatrix}$$

$\alpha = 0$

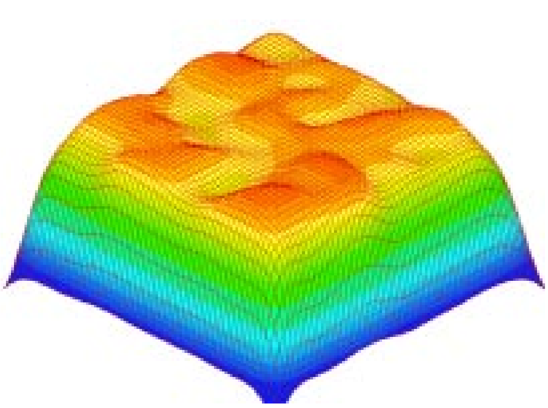
$$\begin{pmatrix} & -1 & \\ -1 & 4 & -1 \\ & -1 & \end{pmatrix}$$

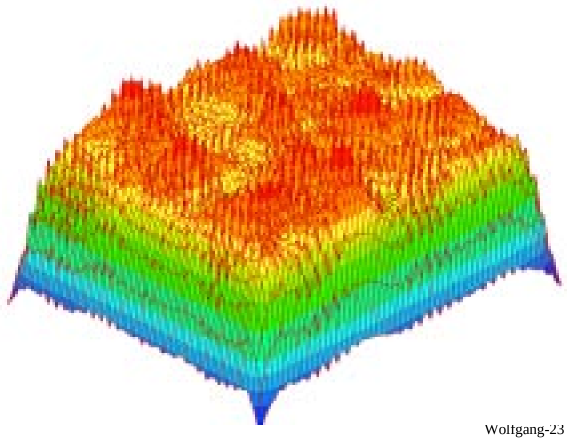
$\alpha = 1/4$

$$\begin{pmatrix} -1 & -1 & -1 \\ -1 & 8 & -1 \\ -1 & -1 & -1 \end{pmatrix}$$

$\alpha = 1/2$

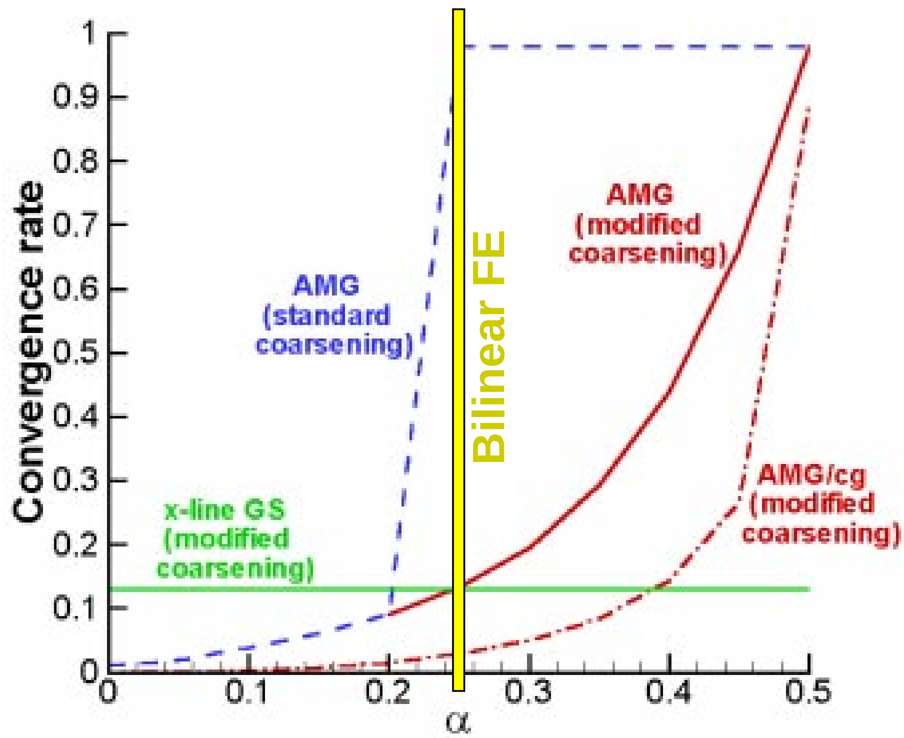
$$\begin{pmatrix} -1 & & -1 \\ & 4 & \\ -1 & & -1 \end{pmatrix}$$





Anisotropic 9-point Case

$$-\varepsilon u_{xx} - u_{yy} = f \quad (\varepsilon \approx 0)$$



Anisotropic 9-point Case

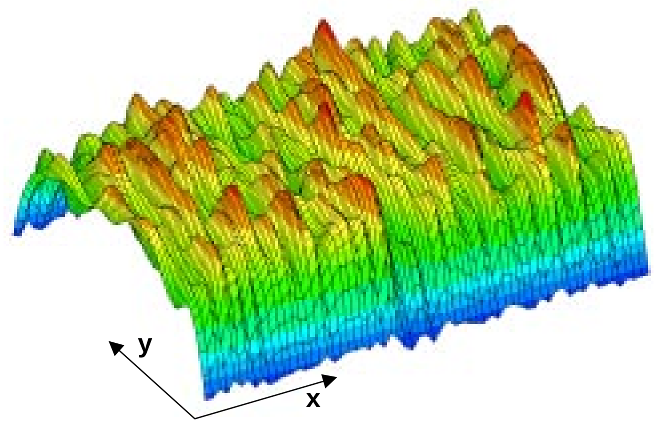
Point relaxation smoothes the error in y-direction.
AMG just does not detect it properly!

$\alpha = 0$

$$\begin{pmatrix} -1 \\ 2 \\ -1 \end{pmatrix}$$

$\alpha = 1/4$

$$\begin{pmatrix} -\frac{1}{4} & -1 & -\frac{1}{4} \\ \frac{1}{2} & 2 & \frac{1}{2} \\ -\frac{1}{4} & -1 & -\frac{1}{4} \end{pmatrix}$$



Modified definition of strong connections

Elimination of positive couplings:

$$\begin{pmatrix} -\frac{1}{4} & -1 & -\frac{1}{4} \\ \frac{1}{2} & 2 & \frac{1}{2} \\ -\frac{1}{4} & -1 & -\frac{1}{4} \end{pmatrix}$$

$$\Downarrow$$

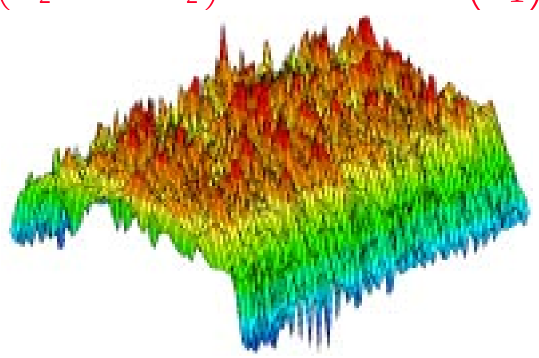
$$\begin{pmatrix} \frac{1}{14} & 0 & -1 & 0 & \frac{1}{14} \\ -\frac{2}{14} & 0 & 2 & 0 & -\frac{2}{14} \\ \frac{1}{14} & 0 & -1 & 0 & \frac{1}{14} \end{pmatrix}$$

Anisotropic 9-point Case

Algebraically smooth error is either (geometrically) smooth in y-direction or highly oscillating in x-direction!

$\alpha = 1/2$

$$\begin{pmatrix} -\frac{1}{2} & -1 & -\frac{1}{2} \\ 1 & 2 & 1 \\ -\frac{1}{2} & -1 & -\frac{1}{2} \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & 1 & \frac{1}{2} \end{pmatrix} \begin{pmatrix} -1 \\ 2 \\ -1 \end{pmatrix}$$



$$\begin{pmatrix} -\frac{1}{2} & -1 & -\frac{1}{2} \\ 1 & 2 & 1 \\ -\frac{1}{2} & -1 & -\frac{1}{2} \end{pmatrix}$$

$$\Downarrow$$

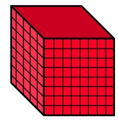
$$\begin{pmatrix} \frac{1}{2} & 0 & -1 & 0 & \frac{1}{2} \\ -1 & 0 & 2 & 0 & -1 \\ \frac{1}{2} & 0 & -1 & 0 & \frac{1}{2} \end{pmatrix}$$

?

AMG with pointwise smoothing cannot work any more!
(x-line relaxation required)

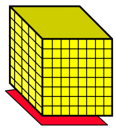
Standard Interpolation, V-cycle

2D: hx=hy=1/128;
3D: hx=hy=hz=1/32



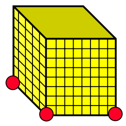
Dirichlet: all around

$\rho^{2D}(V) = 0.265$
$\rho^{3D}(V) = 0.195$



Dirichlet: 1 side

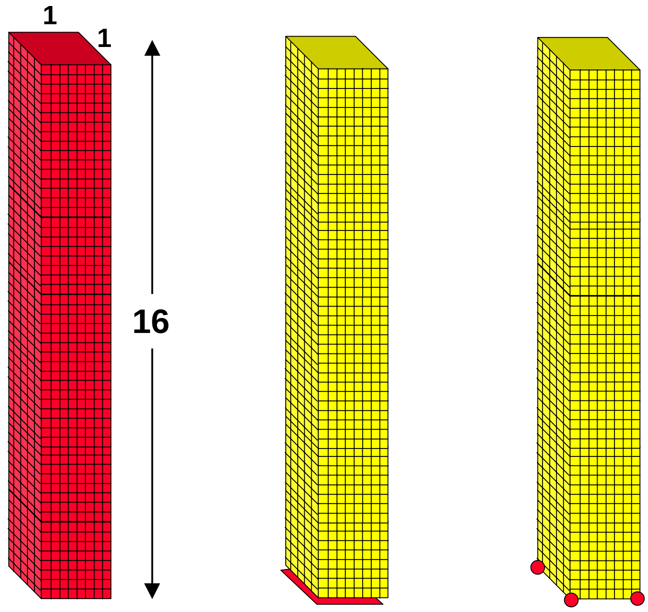
$\rho^{2D}(V) = 0.521$
$\rho^{3D}(V) = 0.711$



Dirichlet: 4 points

$\rho^{2D}(V) = 0.701$
$\rho^{3D}(V) = \underline{0.983}$

2D: hx=hy=1/32;
3D: hx=hy=hz=1/16



$\rho^{2D}(V) = 0.258$	$\rho^{2D}(V) = \underline{0.98}$	$\rho^{2D}(V) = \underline{0.97}$
$\rho^{3D}(V) = 0.168$	$\rho^{3D}(V) = \underline{0.98}$	$\rho^{3D}(V) = \underline{0.99}$

Standard Interpolation, V-cycle

Reason for slow convergence

Increasingly small first eigenvalues of A

Remedy

Improved interpolation
(RBMs instead of true first eigenvectors)

Strategies

Aggregation based AMG
Testfunction-based interpolation

AMGe
Interpolation based on local stiffness matrices

Here:
Interpolation based on geometrical information
(just knowledge of coordinates)

Improvement of Interpolation

Condition for h-independent two-level convergence

$$||e_F - I_{FC}e_C||_D^2 \le \tau ||e||_A^2 \quad \text{for all} \quad e = (e_F, e_C)^T$$

A posteriori improvement of weights w_{ij} :

$$\min\left\{||e_F - I_{FC}e_C||_D^2 \ : \ e \in \{test\ functions\},\ ||e||_A = 1\right\}$$

Constraint:

$$\sum (w_{ij} - w_{ij}^{old})^2 \ \text{minimal!}$$

In practice: Local least squares fit

$$\{test\ functions\} = \{rigid\ body\ modes\}$$

separately for u, v and w:

$$u \to 1, y, z \qquad v \to 1, x, z \qquad w \to 1, x, y$$

(only in direction of strong couplings!)

Improvement of Interpolation

Remarks on the implementation

- Variables with only 1 strong coupling become C-variable
- Coarsening: first boundary, then the interior
- Least squares fit is done immediately before truncation of interpolation takes place

Test Case: Cantilever

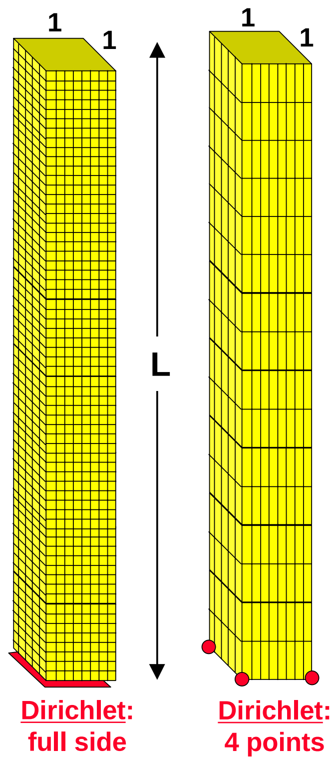
ν = 1/3

Equidistant mesh:

2D:
hx=hy=1/32
(plane strain)

3D:
hx=hy=hz=1/16

L:
1,2,4,8,16,32



Large aspect ratio:

2D:
hx=1/128, hy=L*hx
(plane strain)

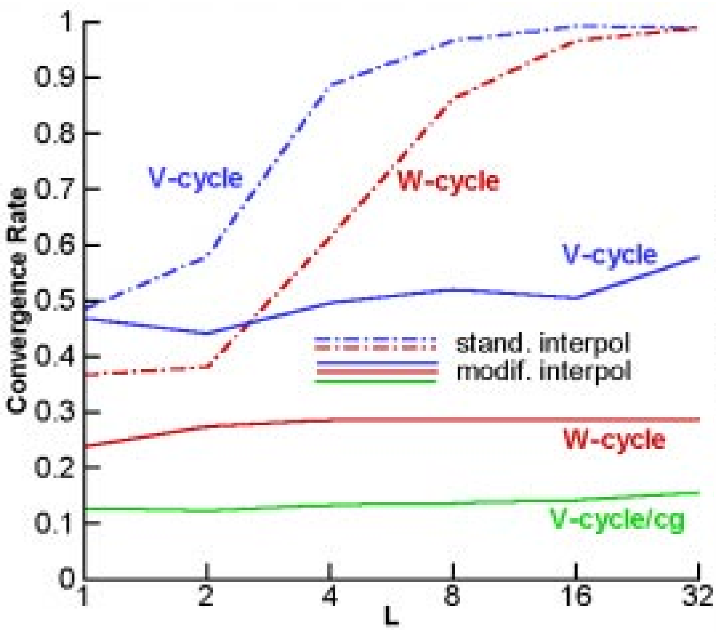
3D:
hx=hy=1/32, hz=L*hx

L:
1,2,4,8,16,32,64,128

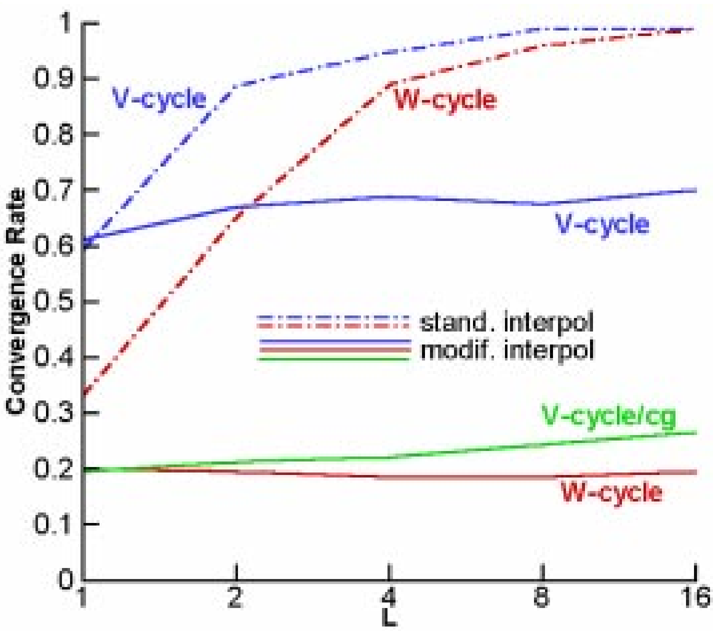
Cantilever

Equidistant mesh case

2D, plain strain



3D

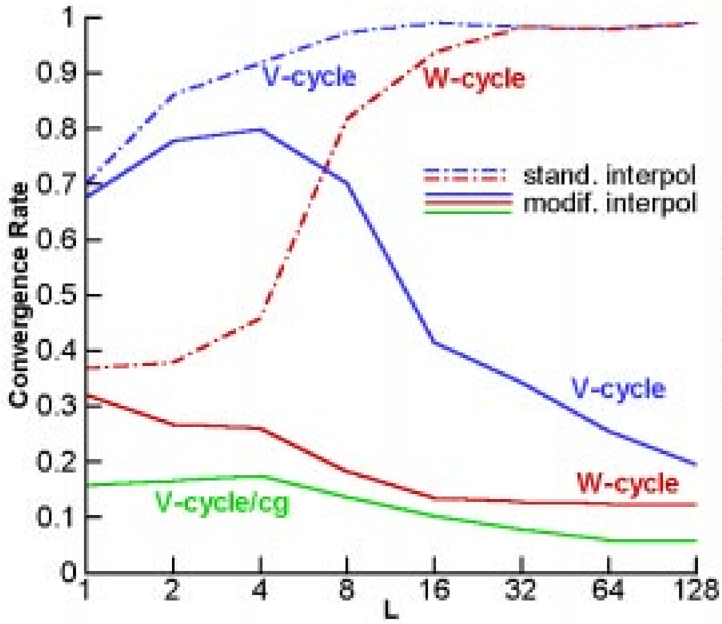


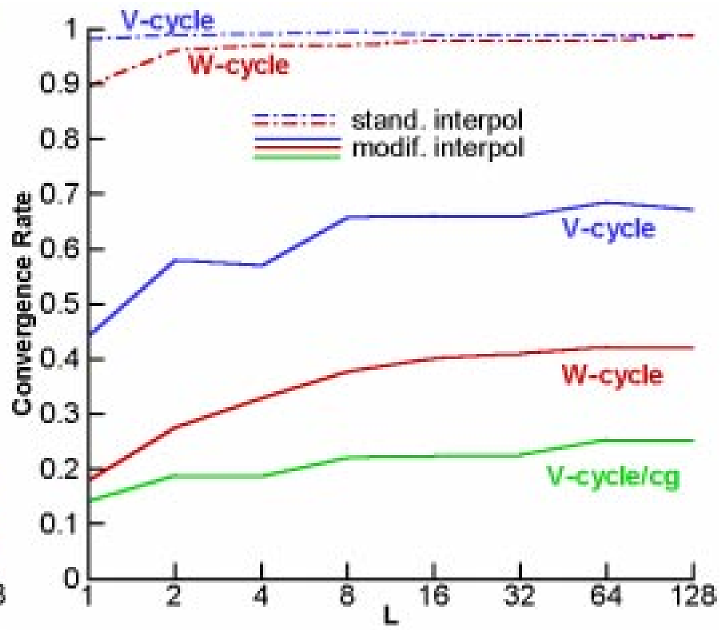
Cantilever

Large aspect ratio case

2D, plain strain
(anisotropy in 1D)

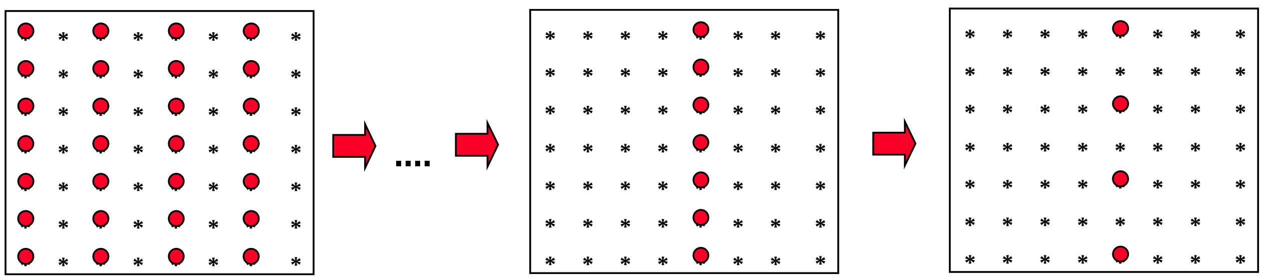
3D
(anisotropy in 2D)



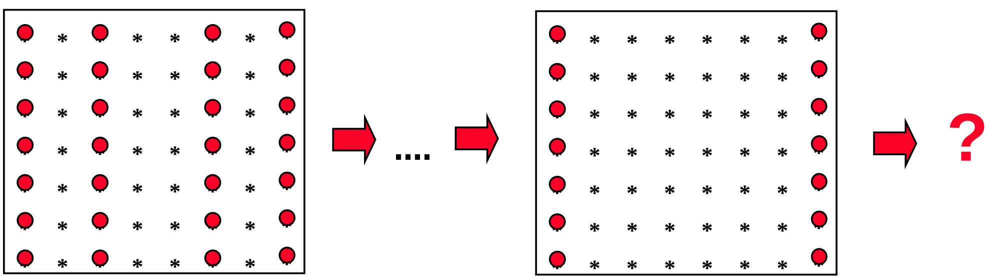


1-Dimensional Anisotropy

Coarsening & standard interpolation



Coarsening & improved interpolation



Remedy:

Block relaxation

Conclusions

- Modified AMG can cope with
  - large aspect ratios
  - any combination of fixed/free boundary conditions
  - RBMs are treated sufficiently well
- This required
  - modified definition of “strong connectivity”
  - improved interpolation (RBMs)
  - geometric information (point locations)
- Current development
  - reduction of AMG’s complexity (3D!)
    (eg, exploit coordinates, aggressive coarsening)
  - replacement of Least Squares fit
    (relative sensitive to scaling factors, too expensive)
  - test & optimization for complex geometries
    (to which extent do we really have to improve interpolation?)

Conclusions

Conservation law of difficulties:

The total number of difficulties in trying to solve complex problems remains constant.