A DUAL-PRIMAL FETI METHOD FOR THREE-DIMENSIONAL INCOMPRESSIBLE STOKES EQUATIONS

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Abstract. In this paper, a dual-primal FETI method is developed for solving incompressible Stokes equations approximated by mixed finite elements with discontinuous pressures in three dimensions. The domain of the problem is decomposed into non-overlapping subdomains, and the continuity of the velocity across the subdomain interface is enforced by introducing Lagrange multipliers. By a Schur complement procedure, solving the indefinite Stokes problem is reduced to solving a symmetric positive definite problem for the dual variables, i.e., the Lagrange multipliers. This dual problem is solved by a Krylov space method with a Dirichlet preconditioner. At each step of the iteration, both subdomain problems and a coarse problem on a coarse subdomain mesh are solved by a direct method. It is proved that the condition number of this preconditioned dual problem is independent of the number of subdomains and bounded from above by the product of the inverse of the inf-sup constant of the discrete problem and the square of the logarithm of the number of unknowns in the individual subdomain problems. Illustrative numerical results are presented by solving a three-dimensional lid driven cavity problem.

Key words. domain decomposition, Stokes, FETI, dual-primal methods

AMS subject classifications. 65N30, 65N55, 76D07

1. Introduction.

The Dual-Primal Finite Element Tearing and Interconnecting (FETI-DP) methods were first proposed by Farhat et al [8] for elliptic partial differential equations. In this method, the spatial domain is decomposed into non-overlapping subdomains, and the interior subdomain variables are eliminated to form a Schur complement problem for the interface variables. Lagrange multipliers are then introduced to enforce continuity across the interface, except at the subdomain vertices where the continuity is enforced directly, i.e., the neighboring subdomains share the degrees of freedom at the subdomain vertices. A symmetric positive semi-definite linear system for the Lagrange multipliers is solved by using the preconditioned conjugate gradient (PCG) method.

This FETI-DP method has been shown to be numerically scalable for second order elliptic problems if a Dirichlet preconditioner is used. Thus, Mandel and Tezaur [15] have proved that the condition number grows at most as $C(1+log(H/h))^2$ in two dimensions, where H is the subdomain diameter and h is the element size. Klawonn et al [12] proposed new preconditioners of this type and proved that the condition numbers are bounded from above by $C(1+log(H/h))^2$ in three dimensions; these bounds are also independent of possible jumps of the coefficients of the elliptic problem.

In [13], we developed a dual-primal FETI method for the two-dimensional incompressible Stokes problem and proved that the condition number is bounded from above by $C(1+log(H/h))^2$. In this paper, we will extend this algorithm to solving three-dimensional incompressible Stokes problem, give the same condition number bound, and prove the inf-sup stability condition of the coarse level saddle point problem, which appeared as an assumption in [13]

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2. FETI-DP algorithm for Stokes problem. We are solving the following Stokes problem on a three-dimensional, bounded, polyhedral domain Ω,

$$\begin{cases} -\Delta \mathbf{u} + \nabla p &= \mathbf{f}, \quad \text{in } \Omega \\ -\nabla \cdot \mathbf{u} &= 0, \quad \text{in } \Omega \\ \mathbf{u} &= \mathbf{g}, \quad \text{on } \partial\Omega \ , \end{cases} \tag{1}$$

where the boundary velocity $\mathbf{g}$ satisfies the compatibility condition $\int_{\partial\Omega} \mathbf{g} \cdot \mathbf{n} = 0$.

The domain Ω is decomposed into N non-overlapping polyhedral subdomains Ω^i of characteristic size H. The subdomain interface is defined as $\Gamma = (\cup \partial\Omega^i) \backslash \partial\Omega$ and $\Gamma^{ij} = \partial\Omega^i \cap \partial\Omega^j$ is the interface between two neighboring subdomains Ω^i and Ω^j.

Consider subdomain incompressible Stokes problems,

$$\begin{cases} -\Delta \mathbf{u}^i + \nabla p^i &= \mathbf{f}^i, \quad \text{in } \Omega^i \\ -\nabla \cdot \mathbf{u}^i &= 0, \quad \text{in } \Omega^i \\ \mathbf{u}^i &= \mathbf{g}^i, \quad \text{on } \partial\Omega \cap \partial\Omega^i \\ \frac{\partial \mathbf{u}^i}{\partial \mathbf{n}^i} - p^i \mathbf{n}^i &= \lambda^i, \quad \text{on } \Gamma^{ij} \ , \end{cases}$$

$$\begin{cases} -\Delta \mathbf{u}^j + \nabla p^j &= \mathbf{f}^j, \quad \text{in } \Omega^j \\ -\nabla \cdot \mathbf{u}^j &= 0, \quad \text{in } \Omega^j \\ \mathbf{u}^j &= \mathbf{g}^j, \quad \text{on } \partial\Omega \cap \partial\Omega^j \\ \frac{\partial \mathbf{u}^j}{\partial \mathbf{n}^j} - p^j \mathbf{n}^j &= \lambda^j, \quad \text{on } \Gamma^{ij} \ , \end{cases}$$

where $\lambda^i + \lambda^j = 0$. We first form subdomain discrete problems by using an inf-sup stable mixed finite element method on each subdomain. Denote the discrete finite element space for the pressures inside the subdomain Ω^i by Π_I^i, and the subdomain constant pressure space by Π_0. We denote the discrete finite element space for the velocity components on Ω^i by $\mathbf{W}^h(\Omega^i)$, which is decomposed as $\mathbf{W}^h(\Omega^i) = \mathbf{W}_I^i \oplus \mathbf{W}_\Gamma^i$, with $\mathbf{W}_I^i$ the interior velocity part and $\mathbf{W}_\Gamma^i$ the subdomain boundary velocity part. Let $\Pi_I = \prod_{i=1}^N \Pi_I^i$, $\mathbf{W}_I = \prod_{i=1}^N \mathbf{W}_I^i$, and $\mathbf{W}_\Gamma = \prod_{i=1}^N \mathbf{W}_\Gamma^i$ be the corresponding product spaces. $\widetilde{\mathbf{W}}_\Gamma$ is a subspace of $\mathbf{W}_\Gamma$ and is given by

$$\widetilde{\mathbf{W}}_\Gamma = \mathbf{W}_\Pi \oplus \mathbf{W}_\Delta,$$

where the primal subspace $\mathbf{W}_\Pi$ consists of two parts. The first is the subdomain corner velocity part, which is spanned by the nodal finite element basis function $\theta_{\mathcal{V}^{il}}$ at the subdomain corners. The other part corresponds to the integrals of the velocity over each subdomain interface, and it is spanned by the pseudoinverse $\mu_i^\dagger$ of the counting functions μ_i corresponding to each subdomain Ω^i: μ_i is 0 at the interface nodes outside $\partial\Omega^i$ while its value at any node on $\partial\Omega^i$ equals the number of subdomains shared by that node. Its pseudoinverse $\mu_i^\dagger$ is the function $1/\mu_i(x)$ for all interface nodes where $\mu_i(x) \neq 0$, and vanishes at all other points. Also note that, here both μ_i and $\mu_i^\dagger$ vanish at the subdomain corners in our algorithm. $\mathbf{W}_\Delta$ is the dual part, which is the direct sum of the local subspaces $\mathbf{W}_\Delta^i$. In the 3D case,

$$\mathbf{W}_\Delta^i := \{\mathbf{w} \in \mathbf{W}_\Gamma^i : \mathbf{w}(\mathcal{V}^{il}) = 0; \bar{\mathbf{w}}_{\mathcal{F}^{ij}} = 0, \ \forall \mathcal{V}^{il}, \mathcal{F}^{ij} \subset \partial\Omega^i\},$$

with $\bar{\mathbf{w}}_{\mathcal{F}^{ij}}$ defined by

$$\bar{\mathbf{w}}_{\mathcal{F}^{ij}} = \frac{\int_{\mathcal{F}^{ij}} \mathbf{w} d\mathbf{x}}{\int_{\mathcal{F}^{ij}} d\mathbf{x}},$$

where $\mathcal{F}^{ij}$ denotes the faces of the subdomain Ω^i.

With these notations, we can decompose the discrete velocity and pressure space of the original problem (1) as followings

$$\mathbf{W} = \mathbf{W}_I \oplus \mathbf{W}_\Pi \oplus \mathbf{W}_\Delta,$$

$$\Pi = \Pi_I \bigoplus \Pi_0.$$

If we further introduce a Lagrange multiplier space Λ to enforce the continuity of the velocities across the subdomain interfaces, then we have the following discrete problem: find a vector $(\mathbf{u}_I, p_I, \mathbf{u}_\Pi, p_0, \mathbf{u}_\Delta, \lambda) \in (\mathbf{W}_I, \Pi_I, \mathbf{W}_\Pi, \Pi_0, \mathbf{W}_\Delta, \Lambda)$ such that

$$\begin{pmatrix} A_{II} & B_{II}^T & A_{\Pi I}^T & 0 & A_{\Delta I}^T & 0 \\ B_{II} & 0 & B_{\Pi I} & 0 & B_{\Delta I} & 0 \\ A_{\Pi I} & B_{\Pi I}^T & A_{\Pi\Pi} & B_{\Pi 0}^T & A_{\Delta\Pi}^T & 0 \\ 0 & 0 & B_{\Pi 0} & 0 & 0 & 0 \\ A_{\Delta I} & B_{\Delta I}^T & A_{\Delta\Pi} & 0 & A_{\Delta\Delta} & B_\Delta^T \\ 0 & 0 & 0 & 0 & B_\Delta & 0 \end{pmatrix} \begin{pmatrix} \mathbf{u}_I \\ p_I \\ \mathbf{u}_\Pi \\ p_0 \\ \mathbf{u}_\Delta \\ \lambda \end{pmatrix} = \begin{pmatrix} \mathbf{f}_I \\ 0 \\ \mathbf{f}_\Pi \\ 0 \\ \mathbf{f}_\Delta \\ 0 \end{pmatrix}. \tag{2}$$

It is important to note that the B_Δ matrix here is a scaled matrix with its elements given by $\{0, \pm\sqrt{\mu_i^\dagger}\}$ to put different weights on the face nodes and the edge nodes, unlike in the two-dimensional case where B_Δ is constructed from $\{0, \pm 1\}$ because the edge is the only type of subdomain interface. It follows immediately from the definition of B_Δ that, on each subdomain interface $\bar{\mathcal{F}}^{ij}$,

$$(B_\Delta^T B_\Delta \mathbf{w})^i|_{\bar{\mathcal{F}}^{ij}} = \pm(\mu_i^\dagger(\mathbf{w}^i - \mathbf{w}^j))|_{\bar{\mathcal{F}}^{ij}}\ , \forall \mathbf{w} \in \mathbf{W}_\Gamma. \tag{3}$$

Also note that we are requiring the pressure to be continuous across the subdomain interfaces in our algorithm.

By defining a Schur complement operator $\tilde{S}$ as

$$\begin{pmatrix} A_{II} & B_{II}^T & A_{\Pi I}^T & 0 & A_{\Delta I}^T \\ B_{II} & 0 & B_{\Pi I} & 0 & B_{\Delta I} \\ A_{\Pi I} & B_{\Pi I}^T & A_{\Pi\Pi} & B_{\Pi 0}^T & A_{\Delta\Pi}^T \\ 0 & 0 & B_{\Pi 0} & 0 & 0 \\ A_{\Delta I} & B_{\Delta I}^T & A_{\Delta\Pi} & 0 & A_{\Delta\Delta} \end{pmatrix} \begin{pmatrix} \mathbf{u}_I \\ p_I \\ \mathbf{u}_\Pi \\ p_0 \\ \mathbf{u}_\Delta \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \tilde{S}\mathbf{u}_\Delta \end{pmatrix}, \tag{4}$$

solving linear system (2) is reduced to solving the following linear system

$$\begin{pmatrix} \tilde{S} & B_\Delta^T \\ B_\Delta & 0 \end{pmatrix} \begin{pmatrix} \mathbf{u}_\Delta \\ \lambda \end{pmatrix} = \begin{pmatrix} \mathbf{f}_\Delta^* \\ 0 \end{pmatrix}. \tag{5}$$

By using a further Schur complement procedure, the problem is finally reduced to solving the following linear system with the Lagrange multipliers λ as its variables:

$$B_\Delta \tilde{S}^{-1} B_\Delta^T \lambda = B_\Delta \tilde{S}^{-1} \mathbf{f}_\Delta^*, \tag{6}$$

Our preconditioner is the standard Dirichlet preconditioner, $B_\Delta S_\Delta B_\Delta^T$, with S_Δ defined as

$$\begin{pmatrix} A_{II} & B_{II}^T & A_{\Delta I}^T \\ B_{II} & 0 & B_{\Delta I} \\ A_{\Delta I} & B_{\Delta I}^T & A_{\Delta\Delta} \end{pmatrix} \begin{pmatrix} \mathbf{u}_I \\ p_I \\ \mathbf{u}_\Delta \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ S_\Delta \mathbf{u}_\Delta \end{pmatrix}. \tag{7}$$

We have now formed the preconditioned linear system

$$B_\Delta S_\Delta B_\Delta^T B_\Delta \tilde{S}^{-1} B_\Delta^T \lambda = B_\Delta S_\Delta B_\Delta^T B_\Delta \tilde{S}^{-1} f_\Delta^* \ , \tag{8}$$

which is our FETI-DP algorithm to solve the incompressible Stokes problem (1).

In our paper [13], we have shown that both S_Δ and $\tilde{S}^{-1}$ are symmetric positive definite on the space $\mathbf{W}_\Delta$. Therefore a preconditioned conjugate gradient method, as well as GMRES, can be used to solve equation (8). We note that we need to apply both S_Δ and $\tilde{S}^{-1}$ to a vector in each iteration step. Multiplying S_Δ by a vector requires solving subdomain incompressible Stokes problems with Dirichlet boundary conditions, and multiplying $\tilde{S}^{-1}$ by a vector requires solving a coarse level saddle point problem, as well as subdomain problems. In [13], we made an assumption about the inf-sup stability condition of this coarse level problem. In the next section we will give a proof of this inf-sup stability condition in 3D case. This proof is also valid for the 2D case. We will also give the condition number estimate for 3D case in section 4.

3. Inf-sup stability of the coarse saddle point problem. We know, from the definition (4), that to find a vector $\mathbf{u}_\Delta = \tilde{S}^{-1} \cdot \mathbf{w}_\Delta \in \mathbf{W}_\Delta$, for a given $\mathbf{w}_\Delta \in \mathbf{W}_\Delta$, requires solving the following linear system

$$\begin{pmatrix} A_{II} & A_{\Delta I}^T & B_{II}^T & A_{\Pi I}^T & 0 \\ A_{\Delta I} & A_{\Delta\Delta} & B_{\Delta I}^T & A_{\Pi\Delta}^T & 0 \\ B_{II} & B_{\Delta I} & 0 & B_{\Pi I} & 0 \\ A_{\Pi I} & A_{\Pi\Delta} & B_{\Pi I}^T & A_{\Pi\Pi} & B_{\Pi 0}^T \\ 0 & 0 & 0 & B_{\Pi 0} & 0 \end{pmatrix} \begin{pmatrix} \mathbf{u}_I \\ \mathbf{u}_\Delta \\ p_I \\ \mathbf{u}_\Pi \\ p_0 \end{pmatrix} = \begin{pmatrix} 0 \\ \mathbf{w}_\Delta \\ 0 \\ 0 \\ 0 \end{pmatrix}. \tag{9}$$

In our FETI-DP algorithm, we solve this linear system by a Schur complement procedure. We first solve a coarse level problem

$$\begin{pmatrix} S_\Pi & B_{\Pi 0}^T \\ B_{\Pi 0} & 0 \end{pmatrix} \begin{pmatrix} \mathbf{u}_\Pi \\ p_0 \end{pmatrix} = \begin{pmatrix} \mathbf{f}_\Pi^* \\ 0 \end{pmatrix} \ , \tag{10}$$

and then the independent subdomain problems

$$\begin{pmatrix} A_{II} & A_{\Delta I}^T & B_{II}^T \\ A_{\Delta I} & A_{\Delta\Delta} & B_{\Delta I}^T \\ B_{II} & B_{\Delta I} & 0 \end{pmatrix} \begin{pmatrix} \mathbf{u}_I \\ \mathbf{u}_\Delta \\ p_I \end{pmatrix} = \begin{pmatrix} 0 \\ \mathbf{w}_\Delta \\ 0 \end{pmatrix} - \begin{pmatrix} A_{\Pi I}^T \\ A_{\Pi\Delta}^T \\ B_{\Pi I} \end{pmatrix} \mathbf{u}_\Pi. \tag{11}$$

In (10), S_Π is defined by:

$$A_{\Pi\Pi} - \begin{pmatrix} A_{\Pi I} & A_{\Pi\Delta} & B_{\Pi I}^T \end{pmatrix} \begin{pmatrix} A_{II} & A_{\Delta I}^T & B_{II}^T \\ A_{\Delta I} & A_{\Delta\Delta} & B_{\Delta I}^T \\ B_{II} & B_{\Delta I} & 0 \end{pmatrix}^{-1} \begin{pmatrix} A_{\Pi I}^T \\ A_{\Pi\Delta}^T \\ B_{\Pi I} \end{pmatrix}, \tag{12}$$

which corresponds to a discrete Stokes harmonic extension operator $\mathcal{SH}_\Pi : \mathbf{W}_\Pi \to \prod_{i=1}^N \mathbf{W}^h(\Omega^i)$ defined as: for any given primal velocity $\mathbf{u}_\Pi \in \mathbf{W}_\Pi$, find $\mathcal{SH}_\Pi \mathbf{u}_\Pi \in \prod_{i=1}^N \mathbf{W}^h(\Omega^i)$ and $p_I \in \prod_{i=1}^N \Pi_I^i$ such that on each subdomain $\Omega^i, i = 1, ..., N$,

$$\begin{cases} a(\mathcal{SH}_\Pi \mathbf{u}_\Pi, \mathbf{v}^i) + b(\mathbf{v}^i, p_I^i) & = & 0, & \forall \mathbf{v}^i \in \mathbf{W}^h(\Omega^i) \\ b(\mathcal{SH}_\Pi \mathbf{u}_\Pi, q_I^i) & = & 0, & \forall q_I^i \in \Pi^i \\ \mathcal{SH}_\Pi \mathbf{u}_\Pi & = & \mathbf{u}_\Pi, & \text{in the primal space } \mathbf{W}_\Pi. \end{cases} \tag{13}$$

If we define an inner product $s_\Pi(.,.)$, corresponding to the Schur operator S_Π, on the space $\mathbf{W}_\Pi$ as

$$s_\Pi(\mathbf{u}_\Pi, \mathbf{u}_\Pi) = \mathbf{u}_\Pi^T S_\Pi \mathbf{u}_\Pi = a(\mathcal{SH}_\Pi \mathbf{u}_\Pi, \mathcal{SH}_\Pi \mathbf{u}_\Pi), \quad \forall \mathbf{u}_\Pi \in \mathbf{W}_\Pi, \tag{14}$$

then the matrix form of the coarse problem (10) can be written in the following variation form: find $\mathbf{u}_\Pi \in \mathbf{W}_\Pi$ and $p_0 \in \Pi_0$ such that,

$$\left\{ \begin{array}{lcl} s_\Pi(\mathbf{u}_\Pi, \mathbf{v}_\Pi) + b(\mathbf{v}_\Pi, p_0) & = & < \mathbf{f}_\Pi, \mathbf{v}_\Pi >, \forall \mathbf{v}_\Pi \in \mathbf{W}_\Pi \\ b(\mathbf{u}_\Pi, q_0) & = & 0, \forall q_0 \in \Pi_0. \end{array} \right. \tag{15}$$

We now give an inf-sup stability estimate for this coarse problem. To do this we need the following inf-sup stability estimate, from Pavarino and Widlund [16],

LEMMA 1. *The following saddle point problem*

$$\left\{ \begin{array}{lcl} s_\Gamma(\mathbf{u}_\Gamma, \mathbf{v}_\Gamma) + b(\mathbf{v}_\Gamma, p_0) & = & < \mathbf{f}_\Gamma, \mathbf{v}_\Gamma >, \forall \mathbf{v}_\Gamma \in \widetilde{\mathbf{W}}_\Gamma \\ b(\mathbf{u}_\Gamma, q_0) & = & 0, \forall q_0 \in \Pi_0, \end{array} \right. \tag{16}$$

is inf-sup stable, i.e., there is a constant β_Γ *such that*

$$\sup_{\mathbf{w}_\Gamma \in \widetilde{\mathbf{W}}_\Gamma} \frac{b(\mathbf{w}_\Gamma, q_0)^2}{s_\Gamma(\mathbf{w}_\Gamma, \mathbf{w}_\Gamma)} \geq \beta_\Gamma^2 ||q_0||_{L^2}^2, \quad \forall q_0 \in \Pi_0 \ . \tag{17}$$

Here the inner product $s_\Gamma(.,.)$ *is defined as:*

$$s_\Gamma(\mathbf{w}_\Gamma, \mathbf{w}_\Gamma) = a(\mathcal{SH}_\Gamma \mathbf{w}_\Gamma, \mathcal{SH}_\Gamma \mathbf{w}_\Gamma), \quad \forall \mathbf{w}_\Gamma \in \widetilde{\mathbf{W}}_\Gamma, \tag{18}$$

with $\mathcal{SH}_\Gamma : \widetilde{\mathbf{W}}_\Gamma \to \prod_{i=1}^N \mathbf{W}^h(\Omega^i)$ *defined as the standard Stokes harmonic extension (cf. Pavarino and Widlund [16]).*

We also need the following lemmas. Lemma 2 can be found in Klawonn et al [12] and in Pavarino and Widlund [17], Lemma 3 in Bramble and Pasciak [3].

LEMMA 2. *Define an interpolation operator* $I_{\Gamma\Pi} : \widetilde{\mathbf{W}}_\Gamma \to \mathbf{W}_\Pi$ *by:*

$$I_{\Gamma\Pi}\mathbf{w}(\mathbf{x}) = \sum_{\mathcal{V}^{il}} \mathbf{w}(\mathcal{V}^{il})\theta_{\mathcal{V}^{il}}(x) + \sum_{\mathcal{F}^{ik}} \bar{\mathbf{w}}_{\mathcal{F}^{ik}} \mu_i^\dagger(x), \forall \mathbf{w}(\mathbf{x}) \in \widetilde{\mathbf{W}}_\Gamma. \tag{19}$$

We then have

$$|I_{\Gamma\Pi}\mathbf{w}|_{H^{1/2}(\Gamma)}^2 \leq C(1 + log(H/h))|\mathbf{w}|_{H^{1/2}(\Gamma)}^2, \forall \mathbf{w}_\Gamma \in \widetilde{\mathbf{W}}_\Gamma, \tag{20}$$

where C *is a constant independent of* H *and* h.

LEMMA 3. *There exist constants* C_1 *and* C_2*, such that*

$$C_1\beta^2 s_\Gamma(\mathbf{w}_\Gamma, \mathbf{w}_\Gamma) \leq |\mathbf{w}_\Gamma|_{H^{1/2}(\Gamma)}^2 \leq C_2 s_\Gamma(\mathbf{w}_\Gamma, \mathbf{w}_\Gamma), \forall \mathbf{w}_\Gamma \in \widetilde{\mathbf{W}}_\Gamma, \tag{21}$$

and

$$C_1\beta^2 s_\Pi(\mathbf{w}_\Pi, \mathbf{w}_\Pi) \leq |\mathbf{w}_\Pi|_{H^{1/2}(\Pi)}^2 \leq C_2 s_\Pi(\mathbf{w}_\Pi, \mathbf{w}_\Pi), \forall \mathbf{w}_\Pi \in \mathbf{W}_\Pi, \tag{22}$$

where β *is the inf-sup stability constant of the subdomain Stokes problem, and the inner inner product* $s_\Gamma(.,.)$ *and* $s_\Pi(.,.)$ *are defined in (18) and (14), respectively.*

We now prove the following inf-sup stability estimate for the coarse saddle point problem (15).

THEOREM 1.

$$\sup_{\mathbf{w}_\Pi \in \mathbf{W}_\Pi} \frac{b(\mathbf{w}_\Pi, q_0)^2}{s_\Pi(\mathbf{w}_\Pi, \mathbf{w}_\Pi)} \geq C\beta^2\beta_\Gamma^2(1+log(H/h))^{-1}||q_0||_{L^2}^2, \quad \forall q_0 \in \Pi_0, \tag{23}$$

where β is the inf-sup stability constant of subdomain Stokes problem solver, and β_Γ is the inf-sup stability constant in Lemma 1.

Proof: Given the inf-sup stability estimate in Lemma 1, we know, from Fortin [9], that there exit an interpolant operator $\Pi_\Gamma : H^{1/2}(\Gamma) \to \widetilde{\mathbf{W}}_\Gamma$ satisfying

$$\left\{ \begin{array}{l} b(\Pi_\Gamma \mathbf{w} - \mathbf{w}, q_0) = 0, \forall q_0 \in \Pi_0 \\ s_\Gamma(\Pi_\Gamma \mathbf{w}, \Pi_\Gamma \mathbf{w}) \leq \frac{C}{\beta_\Gamma^2}|\mathbf{w}|_{H^{1/2}(\Gamma)}^2. \end{array} \right. \tag{24}$$

In order to prove (23), we just need to show that there exists an operator $\Pi_\Pi : H^{1/2}(\Gamma) \to \mathbf{W}_\Pi$, such that

$$\left\{ \begin{array}{c} b(\Pi_\Pi \mathbf{w} - \mathbf{w}, q_0) = 0, \forall q_0 \in \Pi_0 \\ s_\Pi(\Pi_\Pi \mathbf{w}, \Pi_\Pi \mathbf{w}) \leq C\frac{1}{\beta^2\beta_\Gamma^2}(1+log(H/h))|\mathbf{w}|_{H^{1/2}(\Gamma)}^2. \end{array} \right. \tag{25}$$

By defining $\Pi_\Pi = I_{\Gamma\Pi} \circ \Pi_\Gamma : H^{1/2}(\Gamma) \to \mathbf{W}_\Pi$, we have

$$\begin{array}{rcl} b(\Pi_\Pi \mathbf{w} - \mathbf{w}, q_0) & = & b(I_{\Gamma\Pi}(\Pi_\Gamma \mathbf{w}) - \mathbf{w}, q_0) \\ & = & b(I_{\Gamma\Pi}(\Pi_\Gamma \mathbf{w}) - \Pi_\Gamma \mathbf{w}, q_0) + b(\Pi_\Gamma \mathbf{w} - \mathbf{w}, q_0) \\ & = & \sum_i q_0^i \int_{\Omega^i} \mathrm{div}(I_{\Gamma\Pi}(\Pi_\Gamma \mathbf{w}) - \Pi_\Gamma \mathbf{w}) \\ & = & \sum_i q_0^i \int_{\partial\Omega^i} (I_{\Gamma\Pi}\mathbf{w}_\Gamma - \mathbf{w}_\Gamma) \cdot \mathbf{n} \\ & = & -\sum_i q_0^i \int_{\partial\Omega^i} \mathbf{w}_\Delta \cdot \mathbf{n} \\ & = & 0 \ . \end{array}$$

At the same time, by using Lemma 2, Lemma 3, and equation (24), we have

$$\begin{array}{rcl} s_\Pi(\Pi_\Pi \mathbf{w}, \Pi_\Pi \mathbf{w}) & = & a(\mathcal{SH}(\Pi_\Pi \mathbf{w}), \mathcal{SH}(\Pi_\Pi \mathbf{w})) \\ & \leq & C\frac{1}{\beta^2}|\Pi_\Pi \mathbf{w}|_{H^{1/2}(\Gamma)}^2 \\ & = & C\frac{1}{\beta^2}|I_{\Gamma\Pi}(\Pi_\Gamma \mathbf{w})|_{H^{1/2}(\Gamma)}^2 \\ & \leq & C\frac{1}{\beta^2}(1+log(H/h))|\Pi_\Gamma \mathbf{w}|_{H^{1/2}(\Gamma)}^2 \\ & \leq & C\frac{1}{\beta^2}(1+log(H/h))s_\Gamma(\Pi_\Gamma \mathbf{w}, \Pi_\Gamma \mathbf{w}) \\ & \leq & C\frac{1}{\beta^2\beta_\Gamma^2}(1+log(H/h))|\mathbf{w}|_{H^{1/2}(\Gamma)}^2. \end{array}$$

Therefore, equation (25) is proved.

□

4. Condition number estimate in the 3D case. The following three lemmas are from Klawonn et al [12].

LEMMA 4. *Let $\theta_{\mathcal{F}^{ij}}(x)$ be the cut-off function on the open face $\mathcal{F}^{ij}$, and let I^h denote the interpolation operator onto the finite element space $\mathbf{W}^h(\Omega^i)$. Then,*

$$||I^h(\theta_{\mathcal{F}^{ij}}\mathbf{u})||_{H_{00}^{1/2}(\mathcal{F}^{ij})}^2 \leq C(1+\log(H/h))^2 \left(|\mathbf{u}|_{H^{1/2}(\mathcal{F}^{ij})}^2 + \frac{1}{H}||\mathbf{u}||_{L_2(\mathcal{F}^{ij})}^2\right) \ , \forall \mathbf{u} \in \mathbf{W}_\Gamma^i \ .$$

LEMMA 5. *Let $\mathcal{E}^{ik}$ be any edge of Ω^i which forms part of the boundary of a face $\mathcal{F}^{ij} \subset \partial\Omega_i$. Then,*

$$||\mathbf{u}||_{L_2(\mathcal{E}^{ik})}^2 \leq C(1+\log(H/h))\left(|\mathbf{u}|_{H^{1/2}(\mathcal{F}^{ij})}^2 + \frac{1}{H}||\mathbf{u}||_{L_2(\mathcal{F}^{ij})}^2\right) \ , \forall \mathbf{u} \in \mathbf{W}_\Gamma^i \ .$$

LEMMA 6. *Let $\theta_{\mathcal{E}^{ik}}$ be the cutoff function associated with the edge $\mathcal{E}^{ik}$. Then,*

$$|I^h(\theta_{\mathcal{E}^{ik}}\mathbf{u})|^2_{H^{1/2}(\partial\Omega^i)} \le C\|\mathbf{u}\|^2_{L_2(\mathcal{E}^{ik})} \ , \ \forall u \in \mathbf{W}^i_\Gamma \ .$$

If we denote $\bar{\mathcal{F}}^{ij} = \mathcal{F}^{ij} \cup (\cup_k \mathcal{E}^{ik})$ as the closed face, and let $\theta_{\bar{\mathcal{F}}^{ij}}(x)$ be the cut-off function on $\bar{\mathcal{F}}^{ij}$, then by using the above three lemmas and noting that

$$I^h(\theta_{\bar{\mathcal{F}}^{ij}}\mathbf{u}) = I^h(\theta_{\mathcal{F}^{ij}}\mathbf{u}) + \sum_{\mathcal{E}^{ik}\in\partial\mathcal{F}^{ij}} I^h(\theta_{\mathcal{E}^{ik}}\mathbf{u}),$$

we have the following lemma,

LEMMA 7.

$$\|I^h(\theta_{\bar{\mathcal{F}}^{ij}}\mathbf{u})\|^2_{H^{1/2}(\mathcal{F}^{ij})} \le C(1+\log(H/h))^2\left(|\mathbf{u}|^2_{H^{1/2}(\mathcal{F}^{ij})} + \frac{1}{H}\|\mathbf{u}\|^2_{L_2(\mathcal{F}^{ij})}\right) \ , \ \forall \mathbf{u} \in \mathbf{W}^i_\Gamma \ .$$

Proof:

$$\begin{aligned}
\|I^h(\theta_{\bar{\mathcal{F}}^{ij}}\mathbf{u})\|^2_{H^{1/2}(\mathcal{F}^{ij})} &= \|I^h(\theta_{\mathcal{F}^{ij}}\mathbf{u}) + \textstyle\sum_{\mathcal{E}^{ik}\in\partial\mathcal{F}^{ij}} I^h(\theta_{\mathcal{E}^{ik}}\mathbf{u})\|^2_{H^{1/2}(\mathcal{F}^{ij})} \\
&\le \|I^h(\theta_{\mathcal{F}^{ij}}\mathbf{u})\|^2_{H^{1/2}_{00}(\mathcal{F}^{ij})} + \textstyle\sum_{\mathcal{E}^{ik}\in\partial\mathcal{F}^{ij}} \|I^h(\theta_{\mathcal{E}^{ik}}\mathbf{u})\|^2_{H^{1/2}(\mathcal{F}^{ij})} \\
&\le \|I^h(\theta_{\mathcal{F}^{ij}}\mathbf{u})\|^2_{H^{1/2}_{00}(\mathcal{F}^{ij})} + C\textstyle\sum_{\mathcal{E}^{ik}\in\partial\mathcal{F}^{ij}} \|\mathbf{u}\|^2_{L_2(\mathcal{E}^{ik})} \\
&\le C(1+\log(H/h))^2\left(|\mathbf{u}|^2_{H^{1/2}(\mathcal{F}^{ij})} + \tfrac{1}{H}\|\mathbf{u}\|^2_{L_2(\mathcal{F}^{ij})}\right) \ .
\end{aligned}$$

□

We now prove the key estimate,

LEMMA 8. *For all $\mathbf{w}_\Delta \in \mathbf{W}_\Delta$, we have,*

$$|B_\Delta^T B_\Delta \mathbf{w}_\Delta|^2_{S_\Delta} \le C\frac{1}{\beta^2}(1+\log(H/h))^2|\mathbf{w}_\Delta|^2_{\tilde{S}},$$

where $C > 0$ is independent of h, H.

Proof: We consider an arbitrary $\mathbf{w}_\Delta \in \mathbf{W}_\Delta$. In order to compute its $\tilde{S}-$norm, we determine the element $\mathbf{w} = \mathbf{w}_\Pi + \mathbf{w}_\Delta \in \widetilde{\mathbf{W}}_\Gamma, \mathbf{w}_\Pi \in \mathbf{W}_\Pi$, with the correct minimal property. Then, by the definition of $\tilde{S}$, $|\mathbf{w}_\Delta|_{\tilde{S}} = |\mathbf{w}|_{s_\Gamma}$. We next note that we can subtract any continuous function from $\mathbf{w}_\Delta$ without changing the values of $B_\Delta^T B_\Delta \mathbf{w}_\Delta$; thus, $B_\Delta^T B_\Delta \mathbf{w} = B_\Delta^T B_\Delta \mathbf{w}_\Delta$.

We introduce the notation $(\mathbf{v}^i)_{i=1,\dots,N} := B_\Delta^T B_\Delta \mathbf{w}$. Then, we have to estimate

$$|B_\Delta^T B_\Delta \mathbf{w}|^2_{S_\Delta} = |B_\Delta^T B_\Delta \mathbf{w}|^2_{s_\Gamma} = \sum_{i=1}^N |\mathbf{v}^i|^2_{s^i_\Gamma}.$$

We can therefore focus on the estimate of the contribution from a single subdomain Ω^i. We know, from (3), that

$$\mathbf{v}^i = (B_\Delta^T B_\Delta \mathbf{w})^i = \pm \sum_{\mathcal{F}^{ij}\subset\partial\Omega^i} (\mu_i^\dagger(\mathbf{w}^i - \mathbf{w}^j))|_{\bar{\mathcal{F}}^{ij}}.$$

Then by noting that $\mathbf{v}^i$ vanishes at the subdomain vertices, we can cut the function $\mathbf{v}^i$ using the functions $\theta_{\bar{\mathcal{F}}^{ij}}$,

$$\mathbf{v}^i = \pm \sum_{\mathcal{F}^{ij}\subset\partial\Omega^i} I^h(\theta_{\mathcal{F}^{ij}}(\mu_i^\dagger(\mathbf{w}^i - \mathbf{w}^j))),$$

and we have to estimate its s_Γ^i-norm. We have, by using Lemma 3 and Lemma 7,

$$\begin{aligned} &\|I^h(\theta_{\bar{\mathcal{F}}^{ij}}(\mu_i^\dagger(\mathbf{w}^i-\mathbf{w}^j)))\|_{s_\Gamma^i}^2 \le \tfrac{1}{\beta^2}\|I^h(\theta_{\bar{\mathcal{F}}^{ij}}(\mu_i^\dagger(\mathbf{w}^i-\mathbf{w}^j)))\|_{H^{1/2}(\mathcal{F}^{ij})}^2 \\ \le\ & C\tfrac{1}{\beta^2}(1+\log(\tfrac{H}{h}))^2\Big(|\mathbf{w}^i-\mathbf{w}^j|_{H^{1/2}(\mathcal{F}^{ij})}^2 + \\ &\qquad + \tfrac{1}{H}\|(\mathbf{w}^i-\bar{\mathbf{w}}^i_{\mathcal{F}^{ij}})-(\mathbf{w}^j-\bar{\mathbf{w}}^j_{\mathcal{F}^{ij}})\|_{L_2(\mathcal{F}^{ij})}^2\Big)\,. \end{aligned}$$

We can estimate this expression by

$$C\frac{1}{\beta^2}(1+\log(\frac{H}{h}))^2\left(|\mathbf{w}^i|_{H^{1/2}(\mathcal{F}^{ij})}^2+|\mathbf{w}^j|_{H^{1/2}(\mathcal{F}^{ij})}^2\right),$$

as desired, by applying a Poincaré inequality. Then, by using Lemma 3 again, we have

$$\|I^h(\theta_{\bar{\mathcal{F}}^{ij}}(\mathbf{w}^i-\mathbf{w}^j))\|_{s_\Gamma^i}^2 \le C\frac{1}{\beta^2}(1+\log(\frac{H}{h}))^2\left(|\mathbf{w}^i|_{s_\Gamma^i}^2+|\mathbf{w}^j|_{s_\Gamma^j}^2\right)\ .$$

$\square$

By using Lemma 8, we can easily prove the following scalability theorem (cf. Klawonn et al [12] and Li [13]):

THEOREM 2. *The condition number of the preconditioned linear system (8) is bounded from above by* $C\frac{1}{\beta^2}(1+\log(H/h))^2$, *where* C *is independent of* h, H.

5. Numerical experiments. In [13], we have given some numerical results to demonstrate the scalability of the FETI-DP algorithm for solving two-dimensional incompressible problems. Here we describe a three-dimensional experiment. We are solving a lid-driven cavity problem with $\Omega=[0,1]\times[0,1]\times[0,1]$, $\mathbf{f}=\mathbf{0}$, $g_x=1, g_y=g_z=0$ on the face $y=1$, and $\mathbf{g}=\mathbf{0}$ elsewhere on the boundary. The scalability of this algorithm can be seen from Figure 1, even though the size of the problems is rather limited because of the limitations of our current computer, a SUN Ultra 10 workstation.

In Figure 2, we are using two-dimensional numerical results to verify that the inf-sup stability condition for the coarse level saddle point problem is consistent with our estimate in Theorem 1. We can see, from the left figure, that β_C has a lower bound which is independent of the number of subdomains, and that $1/{\beta_C}^2$ appears to be a linear function of $log(H/h)$, from the right figure.

FIG. 1. *GMRES iterations counts for the 3D Stokes solver vs. number of subdomains for* $H/h=4$ *(left) and vs.* H/h *for* $4\times4\times4$ *subdomains (right)*

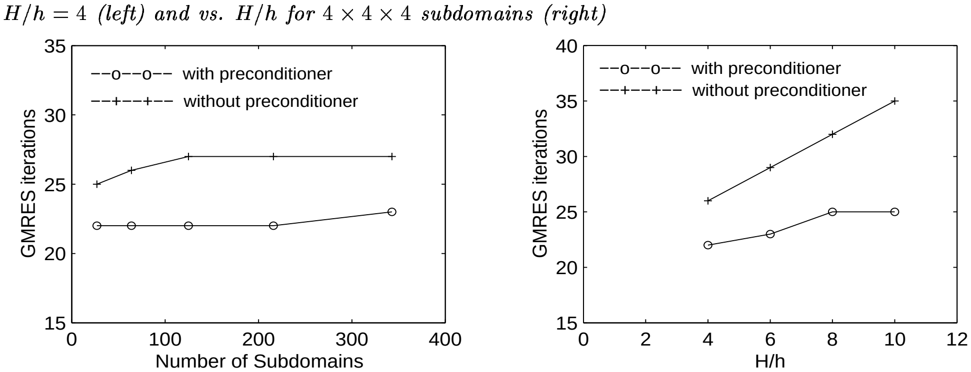
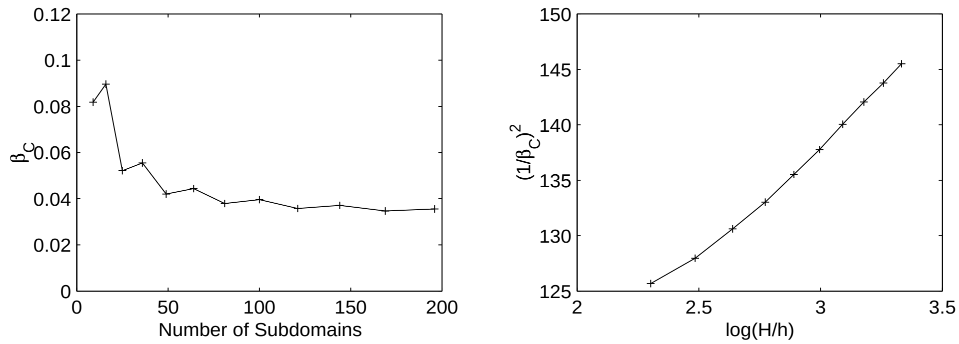


FIG. 2. *Inf-sup stability condition of the coarse level saddle point problem*



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