A NOTE ON SOME ITERATIVE METHODS IN BANACH SPACES

Tchuenche, Jean Michel
Department of Mathematics,
University of Ibadan,
Ibadan, Nigeria.
E-mail: jmtchuenche@yahoo.com

Abstract: After a cogent of iterative methods via contraction mapping principle, the author brilliantly cut to the heart of the question with application to a population dynamics problem. This soon transform the original complex edifice to a mouldering integral form through the theory of semigroup of linear operators.

Key words: successive approximations, population dynamics, abstract spaces.

1. Introduction

This short note starts with a brief survey of the method of successive approximations using fixed point theorem. This technique is used to ascertain the existence and uniqueness of a particular integro-differential equation. The note ends with an application. The main difficulty and restriction is the Lipschitz continuity of the functions, and in a forthcoming article, we will find out if the results still hold without the assumption of Lipschitz continuity.

Theorem 1. Let X be a Banach space and f a contraction on X. Then, there exists a unique point $x_0 \in X \ni f(x_0) = x_0$.

Proof: The proof is straightforward and assumes the Lipschitz continuity of f with $0 < \lambda < 1$ as the Lipschitz constant. If $f(x_0) = x_0$ and $f(y_0) = y_0$, $x_0, y_0 \in X$, then

$$\|x_0 - y_0\| \leq \|f(x_0) - f(y_0)\| \leq \lambda\|x_0 - y_0\|$$

Since $\lambda < 1$, this is only possible if $x_0 = y_0$. Hence, there is at most one fixed point. Now, define $\{x_n\}_{n=1}^{\infty}$ by $x_1 = f(x_0)$, $x_{n+1} = f(x_n)$, $n \geq 1$

$\|x - x_1\| = \alpha$, then

$\|x_{n+1} - x_n\| \leq \lambda^n \alpha$; also for $q > p > 1$

$\|x_q - x_p\| \leq \alpha \dfrac{\lambda^p}{1-\lambda}$. Since $\lambda^p \to 0$ as $p \to \infty$, this shows that $\{x_n\}$ is Cauchy and X being complete, $x_n \to x_0$.

Finally, given $\epsilon > 0$, choose n such that $\|x_0 - x_n\| \leq \dfrac{\epsilon}{2}$ and $\|x_{n-1} - x_n\| \leq \frac{\epsilon}{2}$, so

$$\begin{aligned}
\|x_0 - f(x_0)\| &\leq \|x_0 - x_n\| + \|x_n - f(x_0)\| \\
&\leq \frac{\epsilon}{2} + \|x_n - x_1\| \\
&< \epsilon
\end{aligned}$$

Corollary: Let (M, ρ) be a metric space. If T is a contraction on M, then there is one and only one point x in M such that $T_x = x$.

Proof. Let $x, y \in M$, $\rho(T_x, T_y) \leq \alpha\rho(x, y)$.

Then $\rho(T_x^n,\ T_y^n) \leq \alpha^n \rho(x, y)$ and for $m, n \in N$, $m < n$ and $m = n + p$, we have

$$\rho(x_n, x_m) = \rho(x_n, x_{n+p}) \leq \frac{\alpha^n \rho(x_0, x_1)}{1 - \alpha}$$

This result shows that not only a fixed point exists, but repeated iteration of T will lead from any point of M to an approximation of the fixed point [3].

Below is the appliclation to a nonlinear integro-differential equation.

2. An Example. Consider the differentio-integral equation

$$4\frac{dx}{dt} + \sin x + \int_{t/2}^{t} [1 + x^2(s)] \sin(s) ds = 0 \tag{1}$$

and prove that there is a unique continuous map $x(t)$ on $[0, 1]$ with $x(0) = 0$ and $|x(t)| \leq 1$ which is in fact a solution of (1).

Solution: Equation (1) can be written as

$$\frac{dx}{dt} = f(t, x),$$

and it suffices to show that $f(x,t)$ as defined is jointly Lipschitz continuous in its variables.

Let

$$h(t,x) = \int_{t/2}^{t} [1 + x^2(s)] \sin(s) ds,$$

then $1 + x^2(t)$ is continuous in t, $\sin t$ being continuous. Hence

$$h(t,x) = \int_{t/2}^{t} [1 + x^2(s)] \sin(s) ds$$

is a continuous function of t. Also, since $\sin x$ is continuous in x, it follows that $f(t,x) = -\frac{1}{4}[\sin(x) + h(t,x)]$ is continuous in both t and x in some neighbourhood of $(0,0)$.

A natural choice for the norm is the supremum norm.

Now

$$\begin{aligned}
|f(t,x(t)) - f(t,y(t))| &\leq \frac{1}{4}|\sin x(t) - \sin y(t)| + \frac{1}{4}\int_{t/2}^{t} |y^2(s) - x^2(s)| ds \\
&\leq \frac{1}{4}|\sin y(t) - \sin x(t)| + \frac{1}{2}\int_{t/2}^{t} |y(s) - x(s)| ds, \ s \in [\frac{t}{2}, t] \\
&\leq \frac{1}{4}|\sin y(t) - \sin x(t)| + \frac{1}{2}\sup|y(s) - x(s)| \, |t - \frac{t}{2}| \ \ s \in [0,1]
\end{aligned}$$

By mean-value theorem, if $g(x) = \sin x$, then

$$|g(x) - g(y)| = |g'(\xi)| \, |x - y| \ \text{ for some } \xi \in (x,y)$$

so

$$\begin{aligned}
|f(t,x) - f(t,y)| &\leq \frac{1}{4}\cos\xi |x - y| + \frac{1}{2}\sup|y(s) - x(s)| \\
&\leq \frac{3}{4}\sup|x(s) - y(s)|, \quad |\cos\xi| \leq 1 \\
&\leq \frac{3}{4}\rho(x,y)
\end{aligned}$$

Hence equation (1) has a unique solution $x(t)$.

The above technique is known as Picard's iterative method.

Theorem 2. Let f be continuous and bounded in a region R of the xy-plane containing the point (x_0, y_0), with $|f(x,y)| \leq \beta$ and a, b be any two numbers such that the region R_1 bounded by the vertical lines $x = a$, $x = b$ and the lines through (x_0, y_0) with slopes $\pm\beta$ lies in R. Then the initial value problem $y' = f(x,y)$, $y(x_0) = y_0$ has a unique solution in the interval $[a,b]$, with its graph lying in R_1.

Proof. Let m be a positive integer such that $F^{[m]}$, the m^{th} iterate of F is a contraction on the metric space M. Also, let $y_1, y_2 \in M$, then

$$|[F(y_1)](x) - [F(y_2)](x)| \leq \beta|x - x_0|d(y_1 - y_2).$$

Using this result and the definition of $F^{[2]}(y) = F(F(y))$, we get

$$\begin{aligned} |[F^{[2]}(y_1)](x) - [F^{[2]}(y_2)](x)| &= \left|\int_{x_0}^{x} \{f(t,[F(y_1)](t) - f(t,[F(y_2)](t)\}dx\right| \\ &\leq \beta^2 \left|\int_{x_0}^{x} |t - x_0|dt\right| d(y_1, y_2) \\ &\leq \beta^2 \frac{|x - x_0|^2}{2} d(y_1, y_2) \end{aligned}$$

Continuing in this fashion, we obtain the general formula

$$\left|\left[F^{[m]}(y_1)\right](x) - \left[F^{[m]}(y_2)\right](x)\right| \leq \beta^m \frac{|x - x_0|^m}{m!} d(y_1, y_2)$$

but $\sum_{m=1}^{\infty} \beta^m \frac{(b-a)^m}{m!}$ is a convergent series and hence, $\beta^m \frac{(b-a)^m}{m!}$ is less than 1 for sufficiently large values of m. However, the disadvantage is the requirement that f be a contraction. This is a severe restriction as shown below.

3. Application. Consider the following population dynamics problem.

Find $u(t,a,g) \in L^1([0,A] \times \Omega)$, $t, a \in I\!R^+$, $g \in \Omega$ such that

$$\begin{aligned} & u_t + u_a + G(a)u_g = R(a,g)u \\ & u(0,a,g) = u_0(a,g) \\ & u(t,0,g) = B(t,g) \\ & \text{is } C^1(I\!R^{2+} \times \Omega : I\!R^+) \end{aligned} \tag{3.1}$$

$u(t,a,g)$ denotes the population density at time t, aged a with the additional structure g which could be size, weight or any other attribute that characterises the population under consideration.

In an attempt to solve problem 3.1, it can happen that two characteristics curves intersect and the solution is constrained to take distinct values at the same point. In order to at least ascertain the existence of a unique weak solution, we transform equation 3.1 into an abstract Cauchy problem

$$\begin{aligned} \frac{du(t)}{dt} &= \mathcal{A}u(t) + F(u(t)) \\ u(0) &= u_0 \end{aligned} \tag{3.2}$$

where

$$\mathcal{A}\phi(a,g) := \frac{-\partial\phi(a,g)}{\partial a},\ F(\phi)(a,g) := -(G(a)\frac{\partial}{\partial g} + R(a,g))\phi(a,g)$$

are linear operators.

If $\mathcal{A}$ is the infinitesimal generator of a C_0-semigroup $T(t), t > 0$ such that $\|T(t)u\|_{L^1} \leq \|u\|_{L^1}$, then $T(t)$ is dissipative [2]. By a variation of constant formula, equation (3.2) becomes

$$\begin{aligned} u(t) &= T(t)u_0 + \int_0^t T(t-s)F(u(s))ds,\ t \in \mathbb{R}^+ \\ u(0) &= u_0 \end{aligned}$$

If $u \in D(\mathcal{A}) = \{u \in L^1,\ \frac{du}{da} \in L^1,\ u(0,g) = B(g)\} = W^{1,1}([0,A] \times \Omega)$,

then

Theorem 3.1 If $T(t)$ is dissipative and $F : L^1 \to L^1$ is Lipschitzian, then equation (3.2) has a unique mild solution.

Before proving this theorem, the following lemma will be used in the sequel.

Lemma 3.2. If the first order derivative u_g is continuous in a closed domain and therefore bounded, then F is Lipschitzian.

Proof: Let $u, v \in L^1$ then

$$\begin{aligned} \|F(u(s)) - F(v(s))\| &= \|-G(a)(u_g - v_g) - R(a,g)(u-v)\|_{L^1} \\ &\leq \max(|G(a)|, |R(a,g)|)\|u-v\|_{W^{1,1}} \\ &= C\|u-v\|_{W^{1,1}} \end{aligned}$$

by Sobolev imbedding theorem [2].

Proof of the theorem.

Let M and w be such that $\|T(t)\| \leq Me^{wt}$, define an operator g by

$$(Su)(t) := T(t)u_0 + \int_0^t T(t-s)F(u(s))ds,$$

S is a contraction in $Y := C([0,\tau]; W^{1,1})$ if $u \in Q := \{v \in Y : v(0) = u_0\}$ $v([0,T])$ lies in a neighbourhood of u_0. Then

$$\begin{aligned} \|Su - Sv\|_Y &= \sup \left\| \int_0^t T(t-s)\{F(u(s)) - F(v(s))\}ds \right\| \\ &\leq CMe^{wt}\tau\|u - v\|_Y \to 0 \text{ as } \tau \to 0^+ \end{aligned}$$

It is an easy matter to show that $S(Q) \subset Q$.

Next, applying the above inequality sufficient, we have

$$\|S^n u(t) - S^n v(t)\| \leq \frac{(c\tau Me^{wt})^n}{n!} \sup \|u - v\|_Y$$

For n large enough, S is a contraction and has a unique fixed point in Y. Thus equation (3.1) possesses a unique weak solution on $[0,\tau]$, and since τ is arbitrary, the result follows on the whole of $\mathbb{R}^+$.

It is an easy matter to show that $\|u(t)\| \leq M\|u_0\|e^{(cM+w)t}$, where c is a constant depending on the parameters of the equation, M and w are constants.

Conclusion One major advantage of the fixed point theorem is that, besides showing the uniqueness of solution, it gives a practical method for finding the solution that is, calculation of successive approximations. We wish to emphasize the economy effort which can be achieved by establishing results that can be applied in many situations.

References

1. Adams, R.A. Sobolev spaces, Academic Press, NY, 1975.

2. Goldstein, J.A. Semigroup of Linear Operators and Applications. Oxford University Press, NY, Clarendon Press, 1985.

3. Light, W.A. An introduction to Abstract Analysis. Chapman Hall Mathematics, 71-89, 1990.