COMPARISONS OF UNSTRUCTURED MULTIGRID AS A NON-LINEAR SOLVER A LINEAR SOLVER OR A PRE-CONDITIONER

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INTRODUCTION

- *Investigate Various forms of Multigrid for Non-Linear Problem Solution*
  - *Non-Linear Solver (FAS Multigrid)*
  - *Linear Solver (CS Scheme + Newton Iteration)*
  - *Preconditioner to Newton-Krylov Method*
    - *Linear Preconditioner (CS Scheme)*
    - *Non-Linear Preconditioner (FAS Scheme)*

MOTIVATION

- *Demonstrate Comparative Performance on Several Non-Linear Problems*
  - *Radiation*
  - *Navier-Stokes*
  - *Drive towards Radiation-Hydrodynamics*
- *Provide Insight on Factors Inhibiting Convergence*
- *Provide Insight on Suitability of Methods for Given Problem*
- *Asymptotic Convergence Efficiency Only Criterion*
  - *Assumes in Region of Non-Linear Convergence Basin*
  - *No Memory Considerations (FAS: Low Memory)*

GUIDING PRINCIPLES

- *For Exact (Newton) Linearizations*
  - *Linear and Non-Linear Convergence Equivalent (Asymptotically)*
    - *Theoretical Arguments*
    - *Numerical Evidence*
  - *Efficiency Determined by Relative Cost of Linear vs. Non-Linear Iteration*
- *For Approximate Linearizations*
  - *Relative Numerical (convergence) Efficiency*
  - *Relative Cost per Iteration*
- *Assuming in Non-Linear Ball of Convergence*
  - *Transient, Weakly Non-Linear Problems*

3 TEST PROBLEMS (Two-Dimensional)

- *Radiation Diffusion*
  - *Exact (Newton) Jacobian*
  - *Highly Non-Linear, Expensive Residual*
  - *Transient*
- *Inviscid Transonic Flow (Euler)*
  - *Inexact Jacobian (1st order)*
  - *Moderately Non-Linear, Cheaper Residual*
  - *Steady State*
- *Subsonic Navier-Stokes Flow*
  - *Inexact Jacobian (1st order)*
  - *Highly Anisotropic Mesh*

LINEAR vs. NON-LINEAR SCHEMES

- *To solve :* $\mathbf{R}(\mathbf{w_{exact}}) = \mathbf{0}$ *with Current estimate:* $\mathbf{R}(\mathbf{w}) = \mathbf{r}$

- *FAS (Coarse Grid Equation):*

$$\mathbf{R_H}(w_H) = \mathbf{R_H}(I_h^H w_h) - I_h^H \mathbf{r} \quad \mathbf{R_H}(w_H) - \mathbf{R_H}(I_h^H w_h) = -I_h^H \mathbf{r}$$

- *Newton Linearization (Fine Grid), (Lin MG: Coarse Grid)*

$$\frac{\partial \mathbf{R_h}}{\partial \mathbf{w_h}} \mathbf{\Delta w_h} = -\mathbf{r} \quad \frac{\partial \mathbf{R_H}}{\partial \mathbf{w_H}} \mathbf{\Delta w_H} = -I_h^H \mathbf{r}$$

- *FAS = Finite Difference Jacobian Approximation:*

$$\frac{\partial \mathbf{R_H}}{\partial \mathbf{w_H}} \mathbf{\Delta w_H} = \mathbf{R_H}(w_H) - \mathbf{R_H}(I_h^H w_h)$$

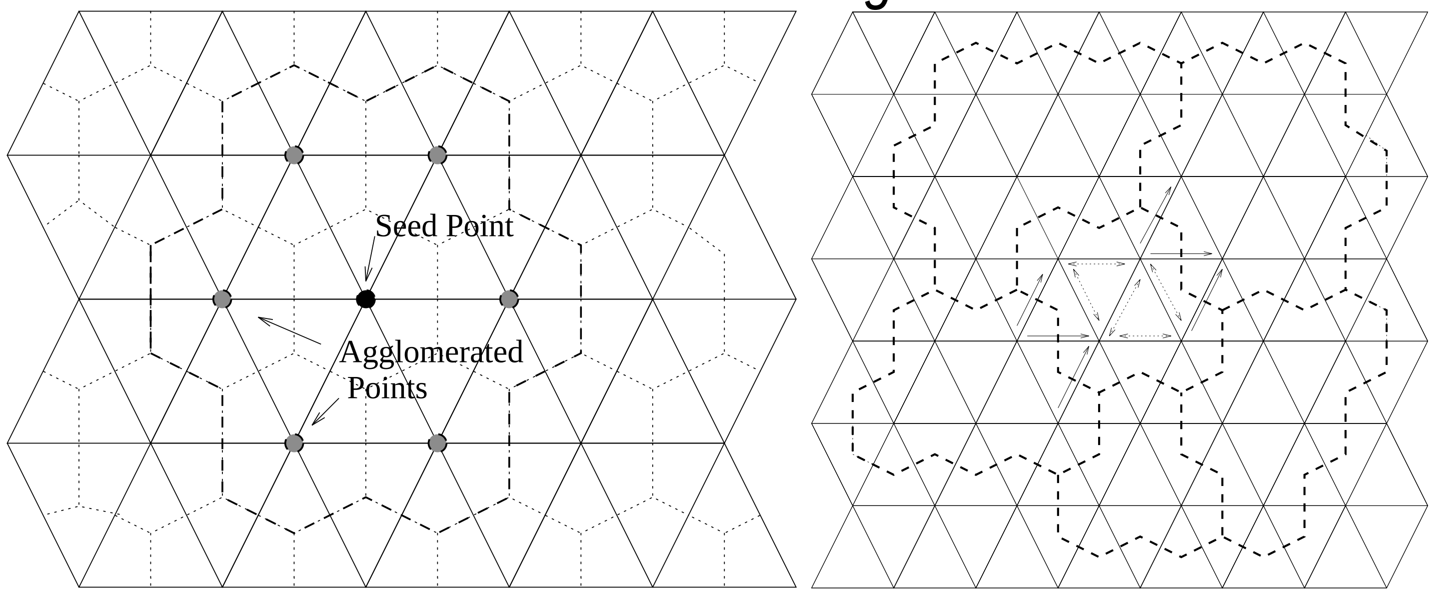
- *Can expect same asymptotic convergence rate (*$\mathbf{\Delta w_H}$ *« 1)*

LINEAR vs. NON-LINEAR SMOOTHING ITERATION

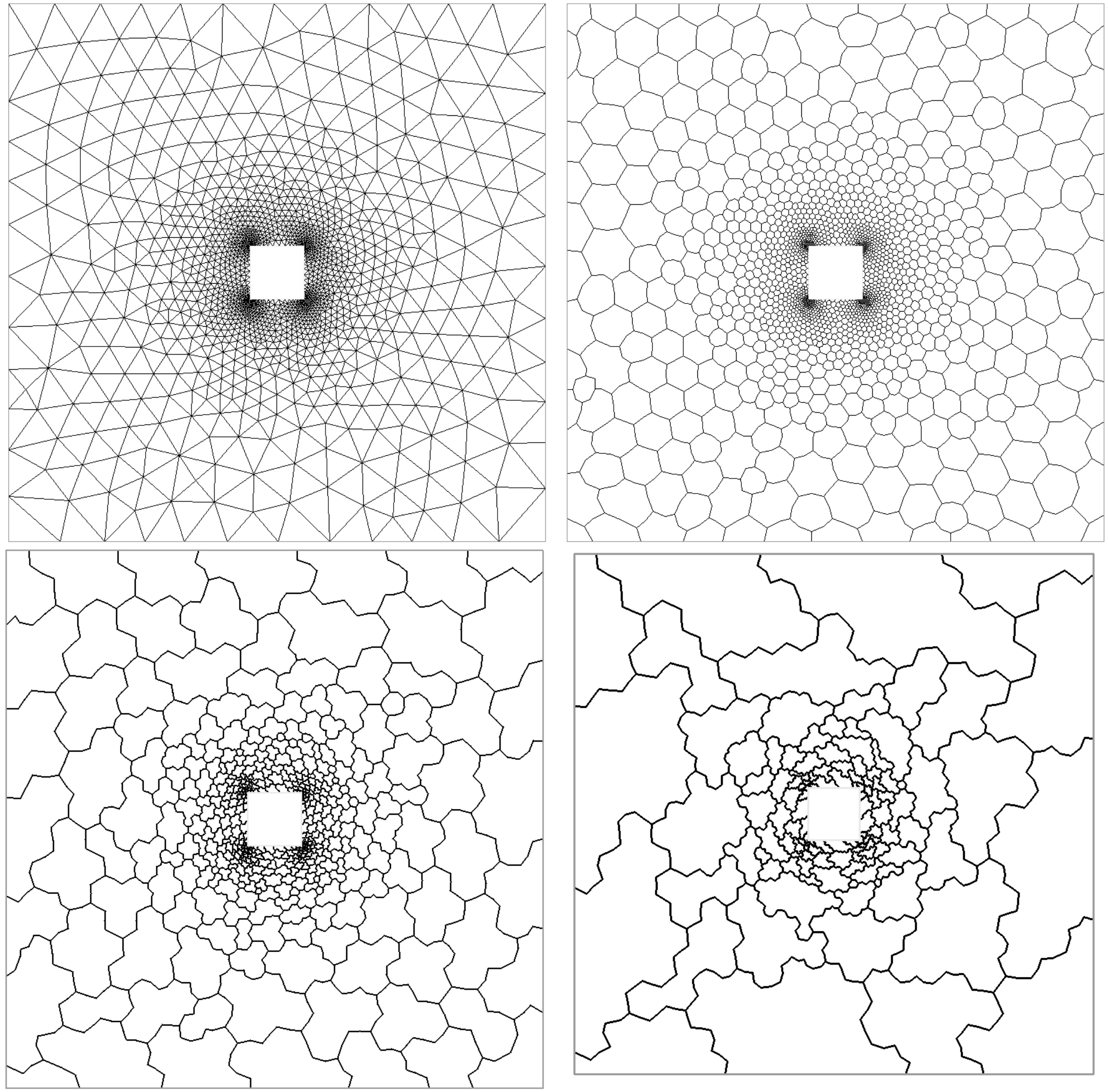
- *Solve :* $\mathbf{R}(\mathbf{w}) = \mathbf{0}$ *or* *Solve :* $\frac{\partial \mathbf{R}}{\partial \mathbf{w}} \mathbf{\Delta w} = -\mathbf{R}(\mathbf{w})$
- *Let:* $\frac{\partial \mathbf{R}}{\partial \mathbf{w}} = [\mathbf{D}] + [\mathbf{O}]$
- *Non-Linear:* $[\mathbf{D}]\mathbf{\Delta w^{n+1}} = -\mathbf{R}(\mathbf{w^n})$
- *Linear :* $[\mathbf{D}]\mathbf{\Delta w^{n+1}} = -\mathbf{R}(\mathbf{w^0}) - [\mathbf{O}]\mathbf{\Delta w^n}$
  - *Jacobi:* $[\mathbf{D}]$ *= Block Diagonal*
  - *Line Solver:* $[\mathbf{D}]$ *= Block Tridiagonal Line Jacobians*
  - *Gauss Seidel: Appropriate Ordering and Updating Strategies*

AGGLOMERATION ALGORITHM

- *Originally Devised for Control Volume Formulations*
- *Reinterpreted as Graph Algorithm*
- *Can be equivalent to AMG given appropriate graph weights*
- *Use Maximal Independent Sets (uniform weights)*
  - *Advancing Front Greedy Algorithm*
- *Piecewise Constant Restriction and Prolongation*



EXAMPLE OF UNWEIGHTED AGGLOMERATION



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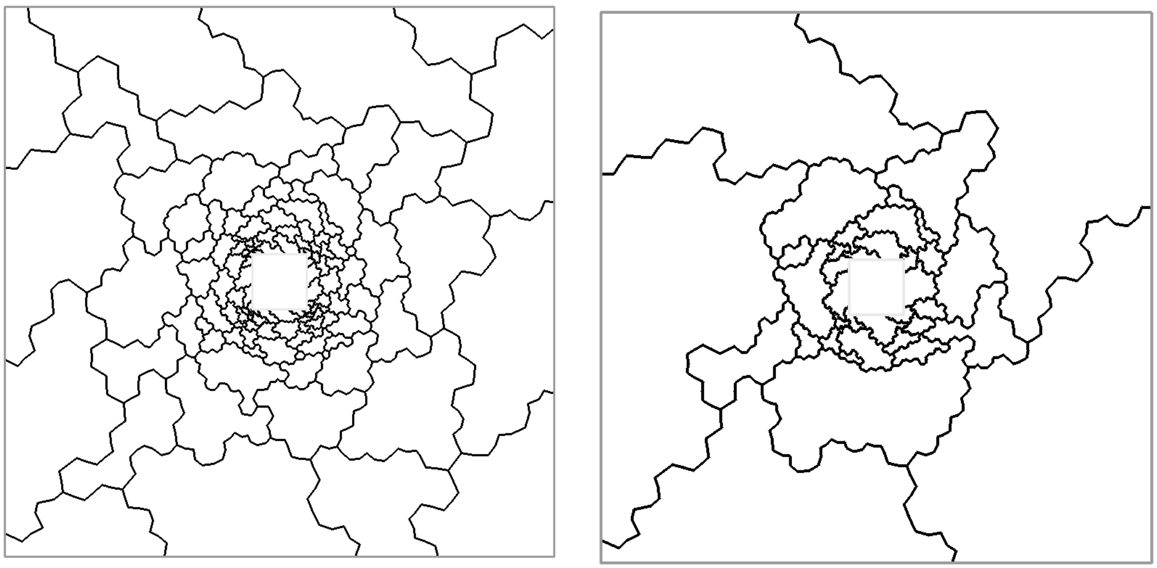


LINEAR vs. NON-LINEAR MULTIGRID

- *Agglomeration MG traditionally implemented as non-linear FAS scheme*
- *Agglomeration can also be used to solve linear system arising at each step of a non-linear Newton solver*
  - *Build fine grid Jacobian, agglomerate to coarse levels*
- *Alternate Linear Multigrid Approach:*
  - *Form Agglomerated Non-Linear Eqns on Each Level (Galerkin: RAP)*
  - *Take Jacobian of these equations on each level*
  - *Solution variables frozen at initial value (within outer non-lin. iter)*
  - *Jacobians updated only at each non-linear update*
  - *More Closely Approximates Linearization of FAS Scheme (for comparison)*

TWO EQUATION RADIATION DIFFUSION MODEL

$$\frac{\partial E}{\partial t} - \nabla.(D_r \nabla E) = \sigma_a (T^4 - E)$$

$$\frac{\partial T}{\partial t} - \nabla.(D_t \nabla T) = -\sigma_a (T^4 - E)$$

with

$$\sigma_a = \frac{z^3}{T^3}, \qquad D_r(T,E) = \frac{1}{3\sigma_a + \frac{1}{E}\frac{\partial E}{\partial x}}, \qquad D_t(T) = \kappa T^{\frac{5}{2}}$$

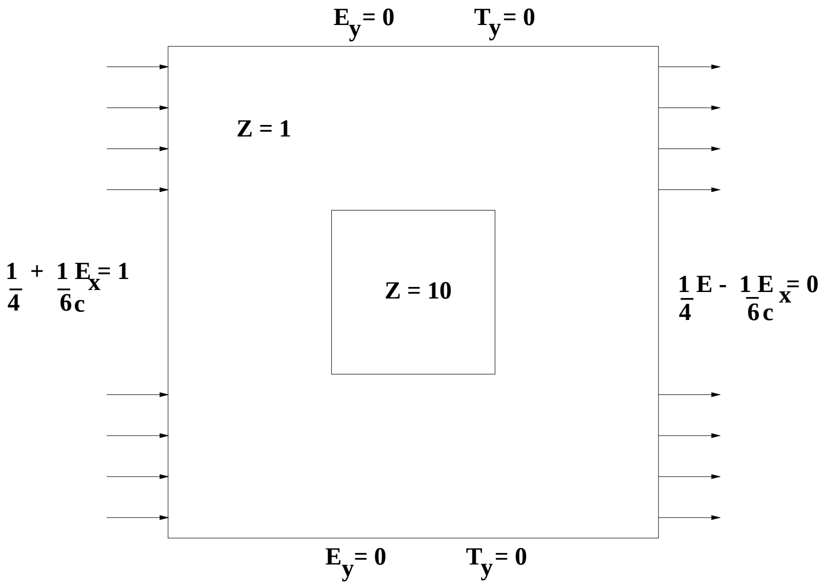
- *Based on Mousseau, Knoll and Rider (LA-UR-99-4230)*
  - *Solve for E, T*
  - *z = atomic number = f(x,y)*

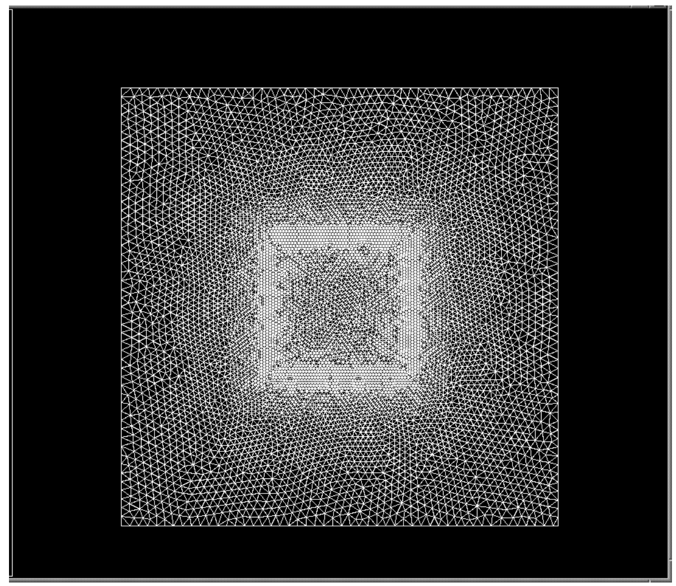
TWO EQUATION RADIATION DIFFUSION MODEL

- *Spatially Discretized as Linear Finite Elements on Triangles*
- *1st Order (backwards Euler) or 2nd Order Time Discretization*
- *Solve Non-Linear Problem at each Time-Step*
  - *FAS Agglomeration MG*
    - *Agglomerate Spatial Coefficients*
    - *Restrict Non-Linear Terms*
  - *Linear MG with outer Non-Linear Iteration*
    - *Linearized Agglomerated Coarse Grid Equations*
    - *Frozen Fine Grid Values throughout non-linear iteration*
- *Compare Accuracy and Performance for Various Approaches*

RADIATION TEST PROBLEM

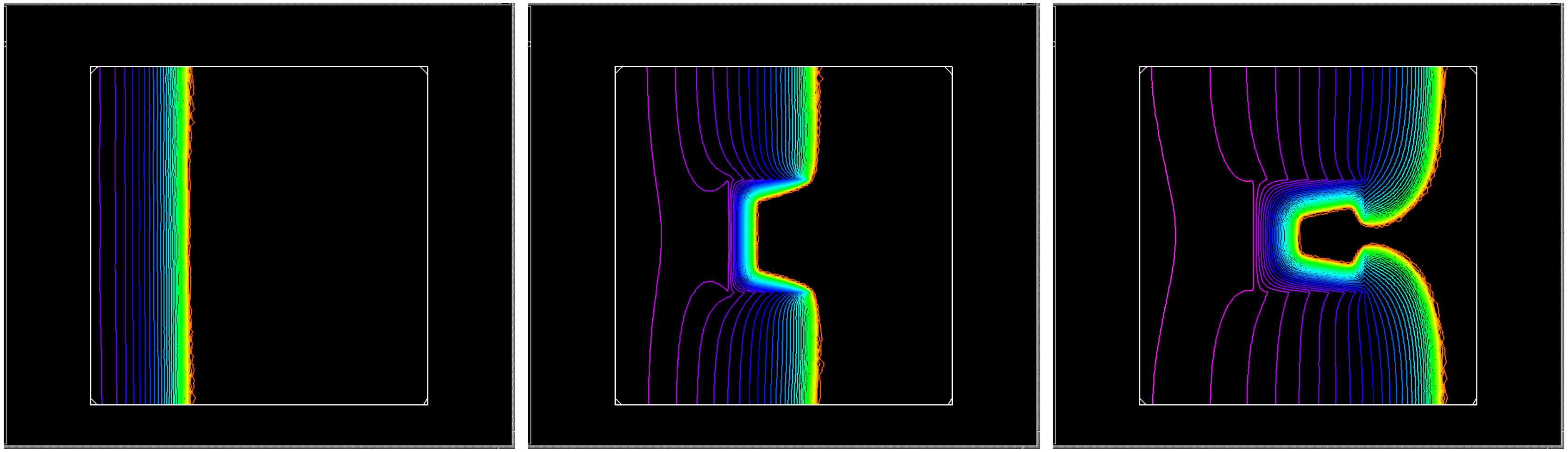
- *Square Region of Inhomogeneous Material*
- *Robin (influx) BC applied at t=0 along x = 0 to initially cold material*
- *Produces jumps in diffusion coefficients of 0(10^6)*
- *Unstructured Grid of 7,502 points, 4 multigrid levels*





RADIATION TEST PROBLEM

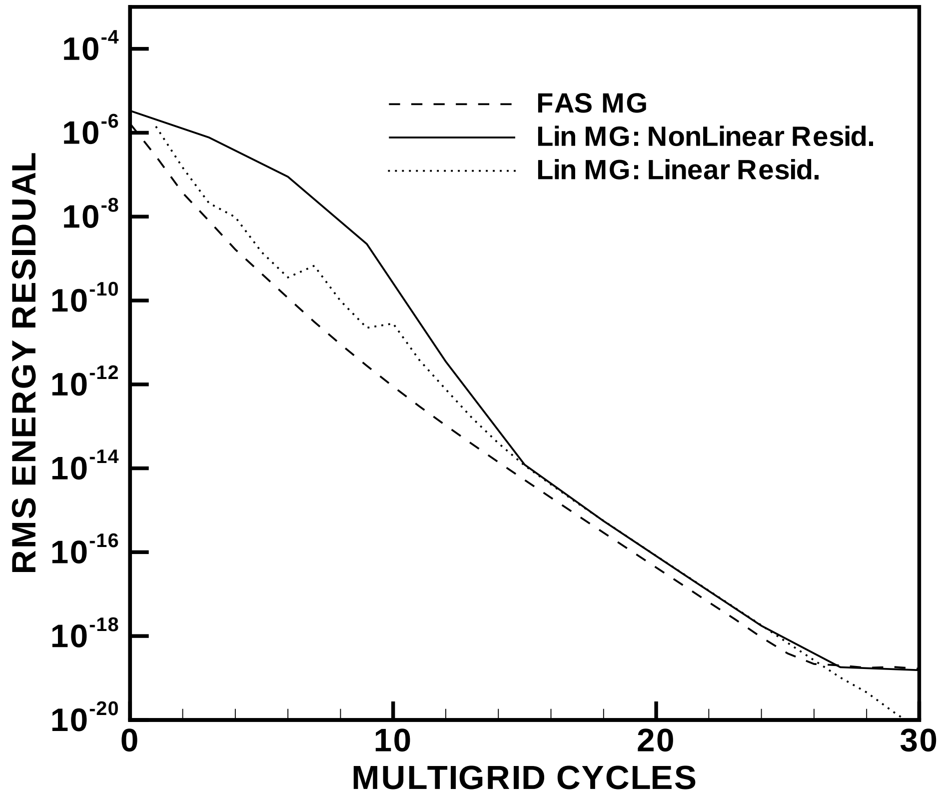
- *Produces Marshak wave, impeded by high Z number region*
- *1st order Backwards Euler time step*
- *Time step = 0.01, Time scale = 10.0*



MULTIGRID SOLUTION OF RADIATION TEST PROBLEM

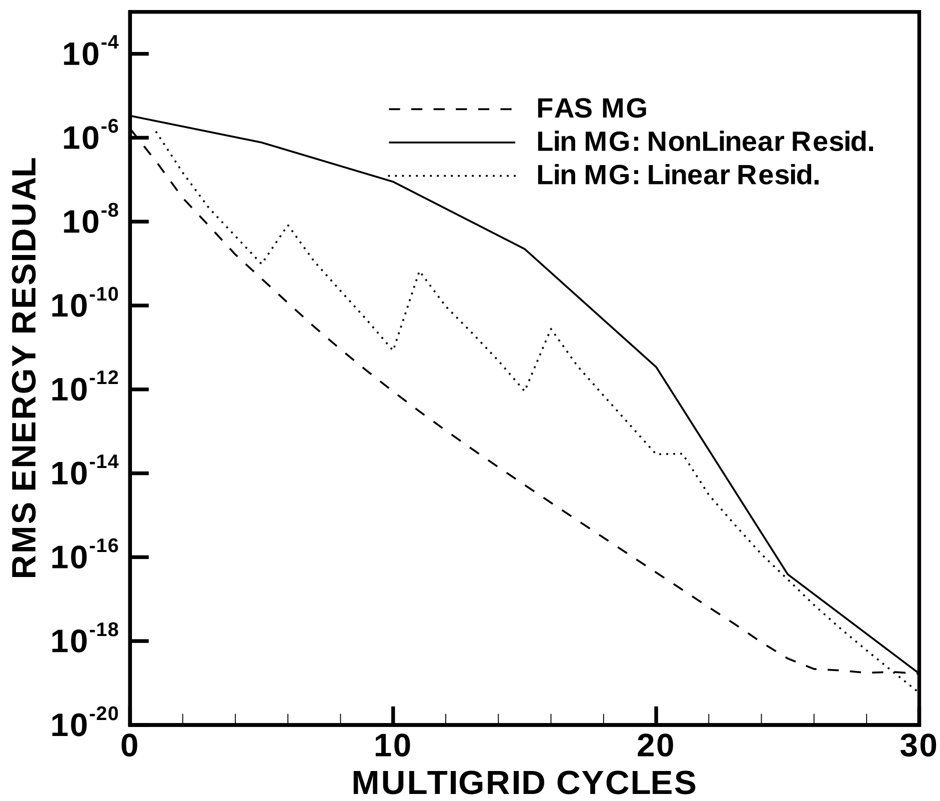
- *Non-Linear Solver: FAS Scheme*
  - *4 Grid Levels, W(3,0)-Cycle*
  - *3 Non-Linear Jacobi Sweeps on each Level*
  - *100 Non-Linear Jacobi Sweeps on Coarsest Level*
- *Linear Solver: CS Scheme + Newton Iteration*
  - *Outer Non-Linear Newton Iteration*
  - *3 Inner W-cycles per Newton Iteration*
  - *4 Grid Levels, W(3,0)-Cycle*
  - *3 Linear Jacobi Sweeps on each Level*
  - *100 Linear Jacobi Sweeps on Coarsest Level*

MG CONVERGENCE RATE FOR 1 PHYSICAL TIME STEP



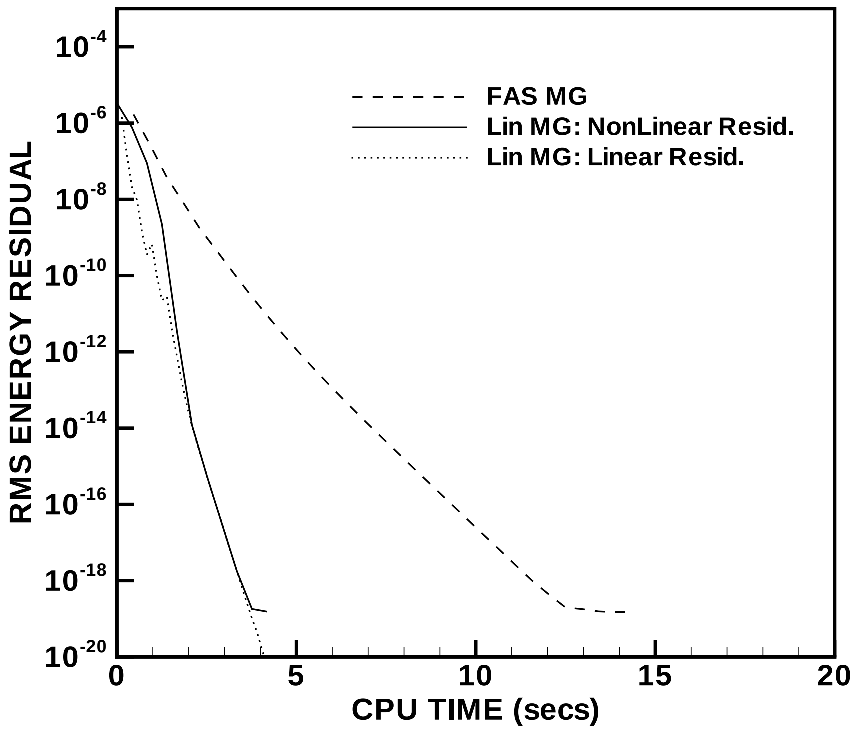
- *Quadratic Convergence of Newton Iteration (Linear MG)*
  - *Becomes limited by insufficient degree of linear system solution*
- *Identical Asymptotic Convergence of FAS and Linear Scheme*
- *Asymptotic Equivalence of Linear and Non-Linear Residuals*

MG CONVERGENCE RATE FOR 1 PHYSICAL TIME STEP



- *Using 5 instead of 3 W-cycles per Newton Iteration*
- *Shift due to Oversolution of Linear System in Initial Phase*

MG CONVERGENCE EFFICIENCY FOR 1 PHYSICAL TIME STEP



Component	Normalized Timing
Non-Linear Residual	1.0
Residual + Point Jacobians	2.52
Residual + Entire Jacobian	2.82
1st Stage Non-Linear Sweep	2.82
Add. Stages Non-Lin. Sweeps	1.07
1st Linear Jacobi Sweep	0.364
Incr. Linear Jacobi Sweeps	0.173
FAS MG Cycle	13.04
Linear MG Cycle	3.31

- *Identical Asymptotic Convergence Rates*
- *Linear Multigrid Sweep Less Costly*
- *Linear Multigrid Method More Efficient*

SOLUTION OF EULER and NAVIER-STOKES EQUATIONS

- *Vertex Discretization on Unstructured Meshes, Roe-Rieman Solver*
- *2nd Order Accuracy through Gradient Reconstruction : Extended Stencil*
- *1st Order Jacobian (typical) : Nearest Neighbor Stencil*
- *FAS : 1st order discretization on coarse grids (typical)*
- *FAS : 1st order local (point and line) Jacobians for precond. R-K Scheme*
- *Linear MG : Avoid oversolving linear system (even in asymptotic region)*
- *Use Multigrid as Preconditioner to Newton-Krylov Method*
  - *Matrix-Free Exact Jacobian*
  - *Linear Preconditioner (CS Scheme)*
  - *Non-Linear Preconditioner (FAS Scheme)*

SOLUTION OF EULER and NAVIER-STOKES EQUATIONS

- *Euler Case*
  - *FAS MG: 3 or 5 stage Jacobi Preconditioned Runge-Kutta Scheme*
  - *Linear MG: 4 linear Jacobi sweeps per grid level*
  - *Linear MG: 4 linear Gauss Seidel sweeps per grid level*
    - *Sweeps alternating in increasing-decreasing x-direction*
- *Navier-Stokes Case*
  - *FAS MG: 3 or 5 stage Line Preconditioned Runge-Kutta Scheme*
  - *Linear MG: 4 linear Line sweeps per grid level*
  - *Linear MG: 4 linear Gauss Seidel Line sweeps per grid level*
    - *Grid points/Lines ordered in x-direction*
    - *Sweeps alternating in increasing-decreasing x-direction*

RELATIVE TIMINGS FOR FAS AND LINEAR MULTIGRID METHODS FOR EULER AND NS PROBLEMS

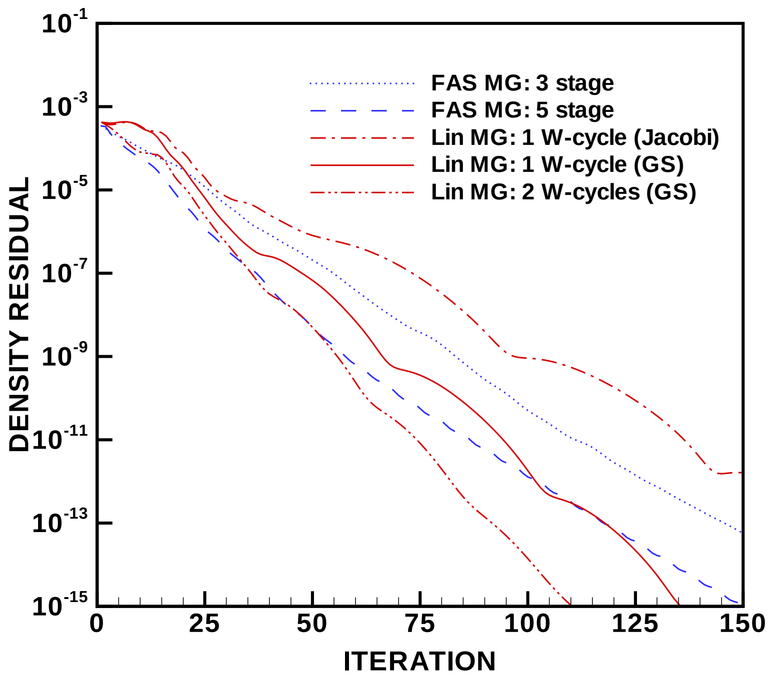
Component (Euler)	Normalized Timing
Non-Linear Residual	1.0
Residual + Line Jacobians	1.62
Residual + Entire Jacobian	1.86
1st Stage Non-Linear Sweep	1.96
Add. Stages Non-Lin. Sweeps	1.26
1st Linear GS Sweep	0.43
Incr. Linear GS Sweeps	0.38
3-stage FAS MG Cycle	8.92
5-stage FAS MG Cycle	9.86
Linear MG Cycle (1 W cycle)	5.70
Linear MG Cycle (2 W cycles)	8.98

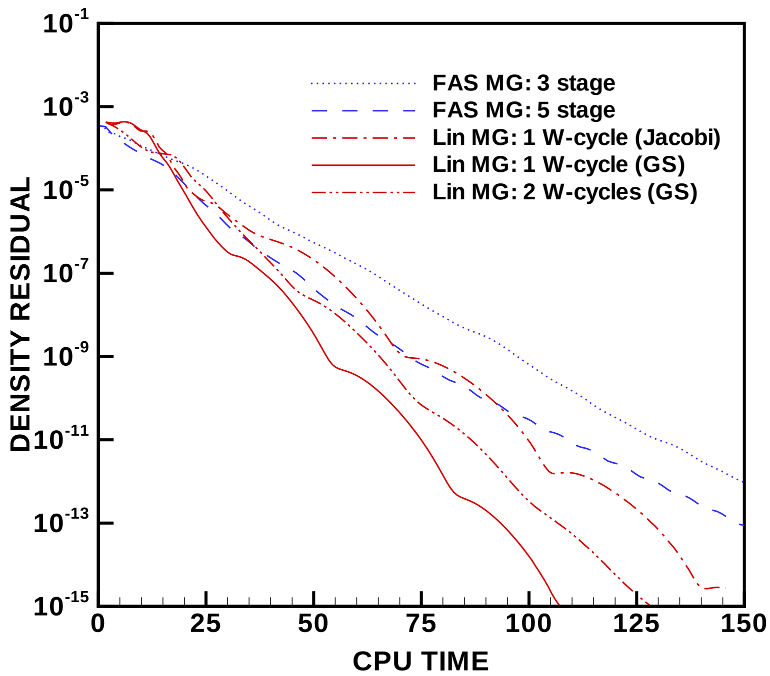
Component (Navier-Stokes)	Normalized Timing
Non-Linear Residual	1.17
Residual + Line Jacobians	2.13
Residual + Entire Jacobian	2.39
1st Stage Non-Linear Sweep	2.44
Add. Stages Non-Lin. Sweeps	1.44
1st Linear GS Sweep	0.43
Add. Linear GS Sweeps	0.38
3-stage FAS MG Cycle	10.4
5-stage FAS MG Cycle	11.3
Linear MG Cycle (1 W cycle)	6.3
Linear MG Cycle (2 W cycles)	9.6

- *Linear Sweeps more Cost Effective*
- *Smaller Ratio than Radiation Problem*

SOLUTION OF EULER EQUATIONS

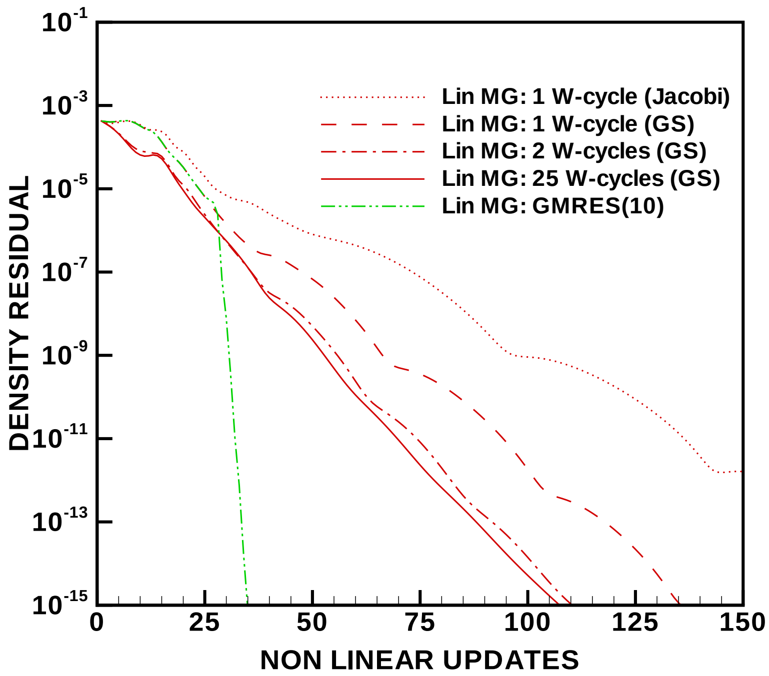
- *Transonic NACA 0012, Mach=0.8, Incidence = 1.25 degrees*
- *5 Level Saw-Tooth W-Cycle, Fine Grid: 7500 points*

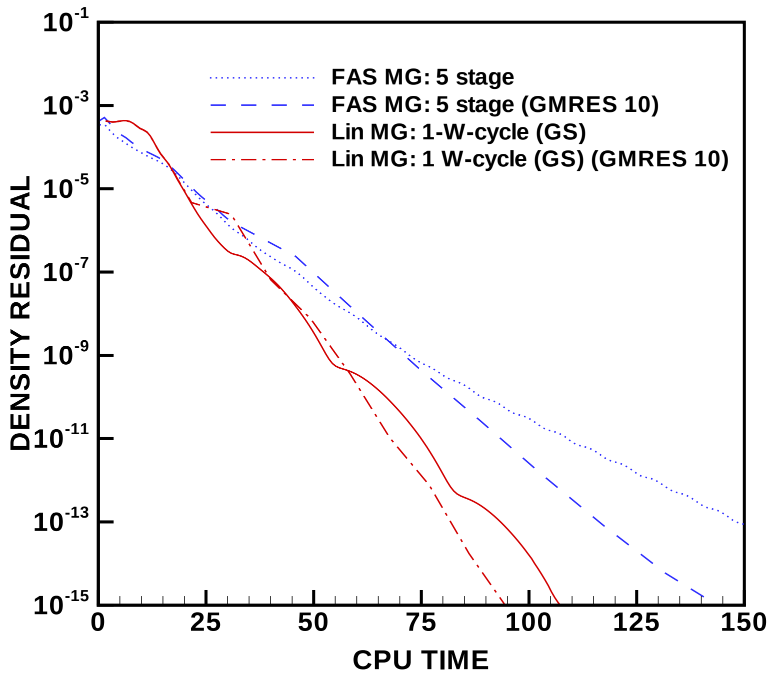




- *Linear Jacobi Slower than Non-Linear Scheme (per cycle)*
- *Single GS W-cycle Most Efficient Overall*
  - *Factor 2 over FAS Schemes due to Cheaper Cycle Cost*

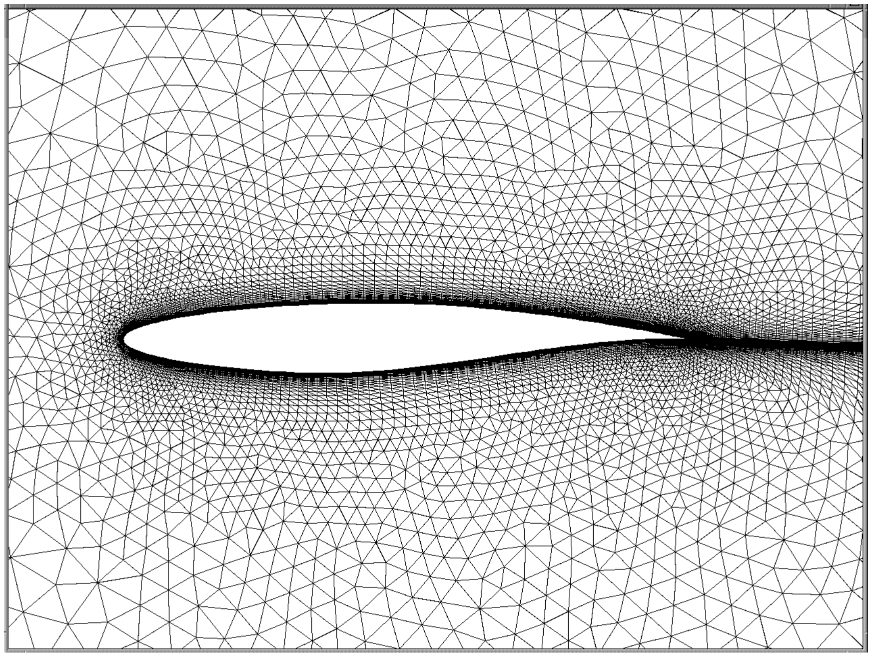
SOLUTION OF EULER EQUATIONS

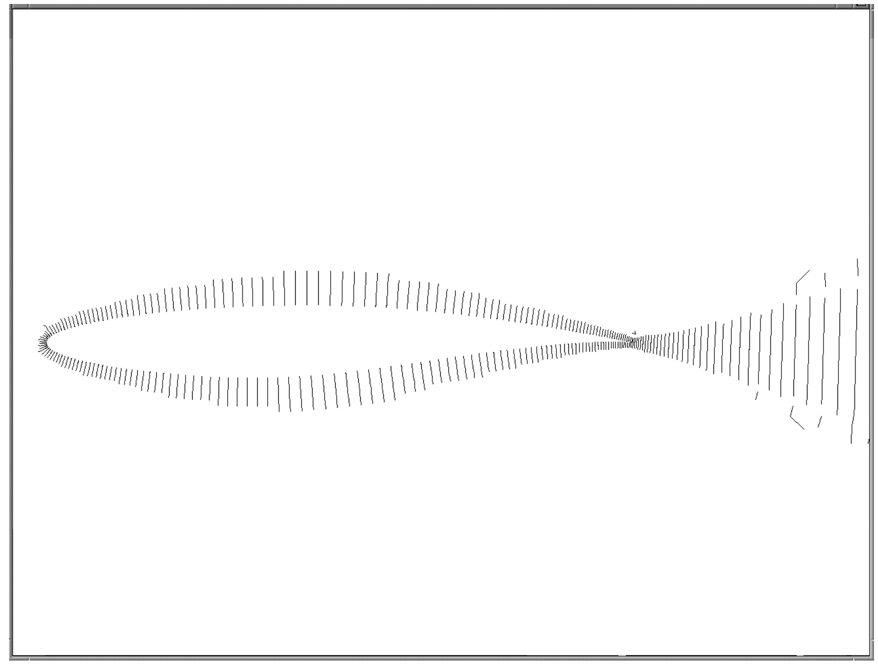




- *Non-Linear Convergence Limited by Inaccurate Jacobian*
- *2 W-cycles provides almost all “usable” linear system convergence*
- *Newton-Krylov Provides Rapid Non-Linear Convergence*
- *Newton-Krylov Yields Modest Gains in Efficiency for Both Linear and Non-Linear Preconditioners*

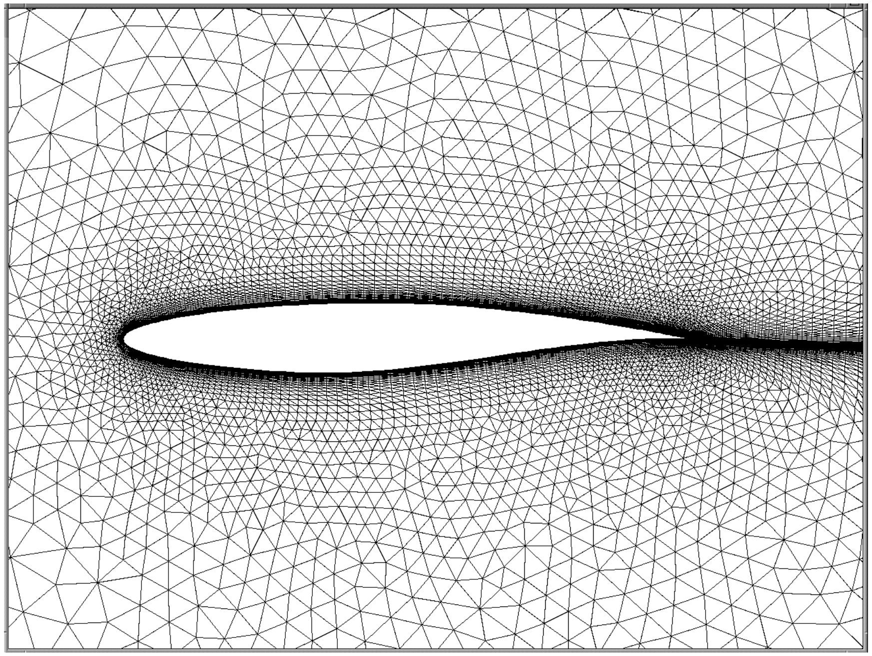
SOLUTION OF VISCOUS AIRFOIL PROBLEM

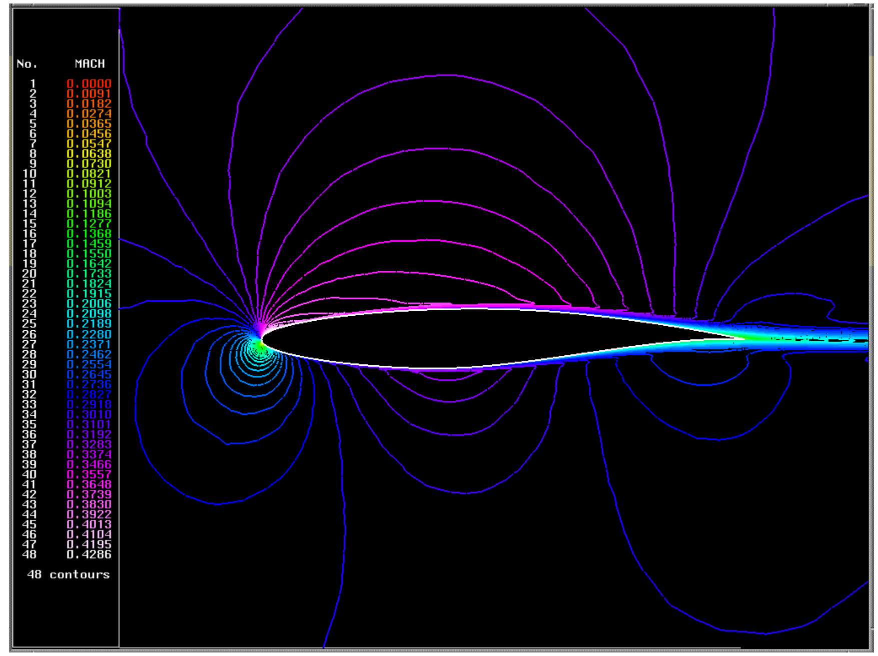




- *Mixed Quad-Triangular Grid: 16,167 points*
- *1.E-06 Spacing at Wall*
- *Mach = 0.3, Incidence = 2.31 degrees, Reynolds Number = 6.5 million*
- *Relatively Easy 2D Navier-Stokes Problems*

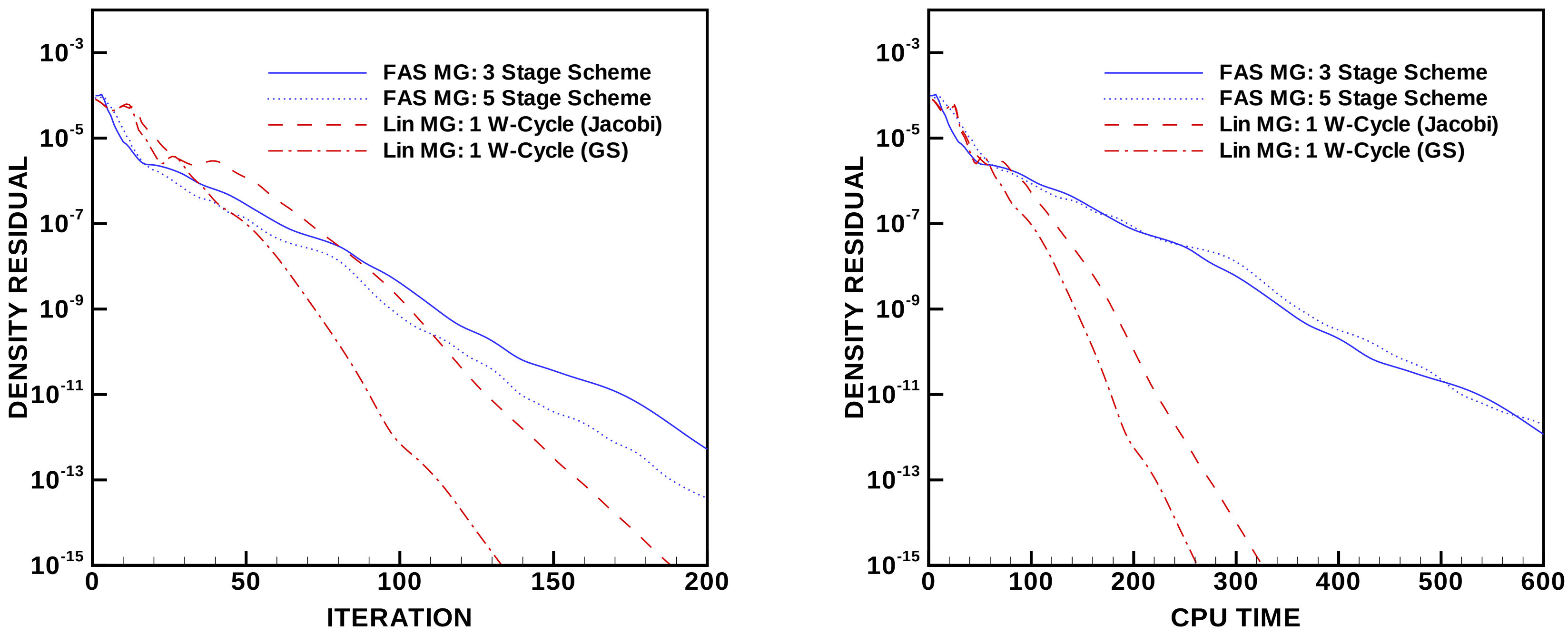
SOLUTION OF VISCOUS AIRFOIL PROBLEM





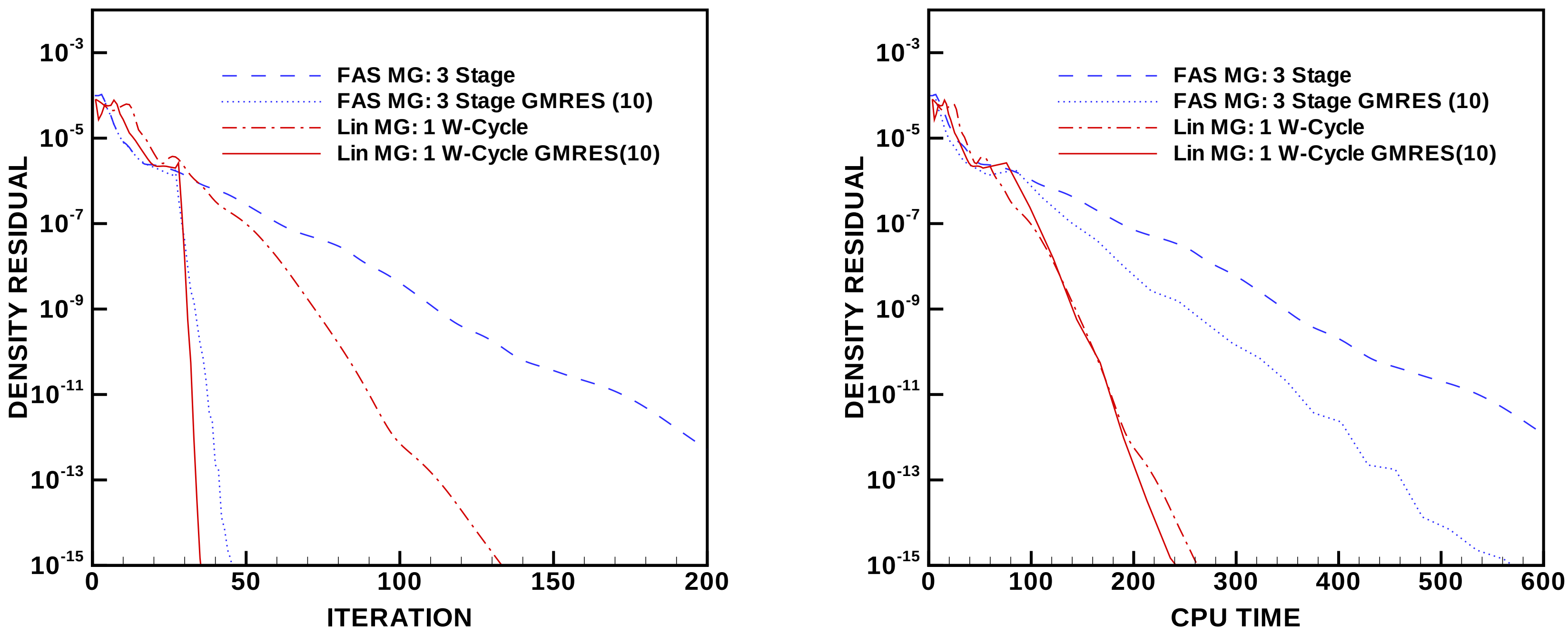
- *Mach = 0.3, Incidence = 2.31 degrees, Reynolds Number = 6.5 million*
- *Spalart-Allmaras Turbulence Model (solved with flow)*
- *Freestream Initial Conditions*

SOLUTION OF VISCOUS AIRFOIL PROBLEM



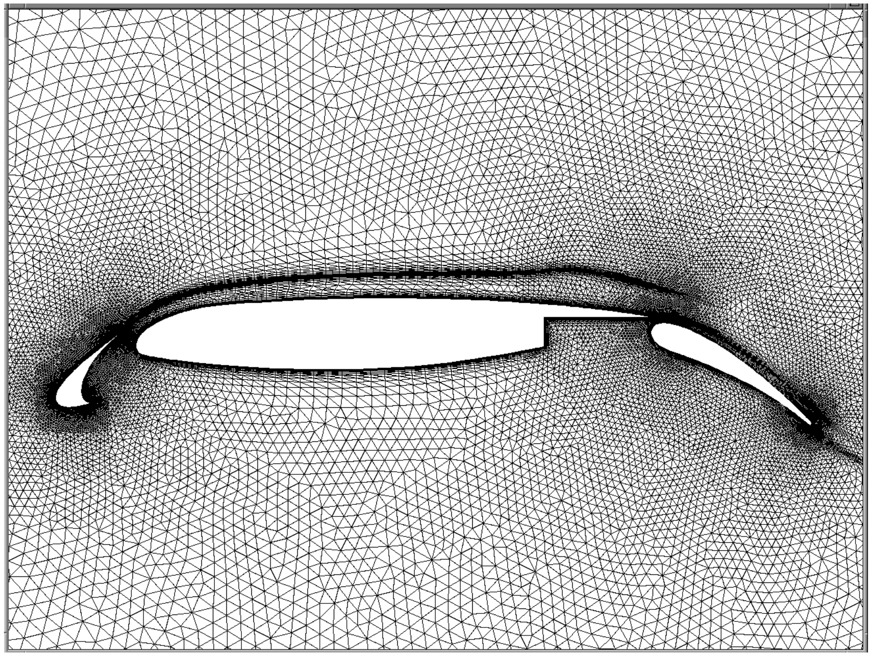
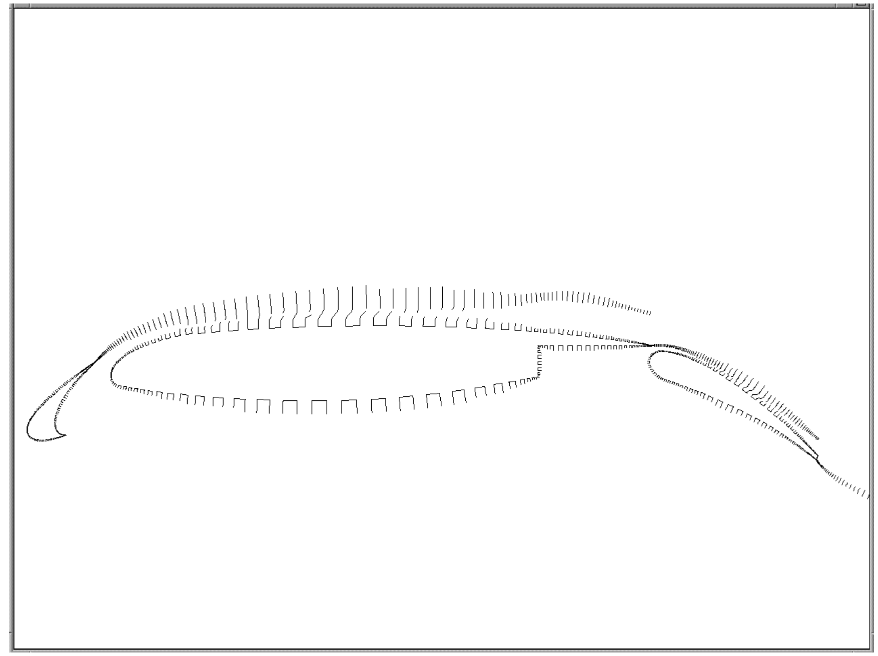
- *Linear Line Jacobi Faster than Non-Linear Line RK-Scheme (per sweep)*
- *1 W-cycle Linear Line Gauss Seidel Most Efficient Strategy*
  - *Factor of 3 Faster then FAS Scheme*

SOLUTION OF VISCOUS AIRFOIL PROBLEM



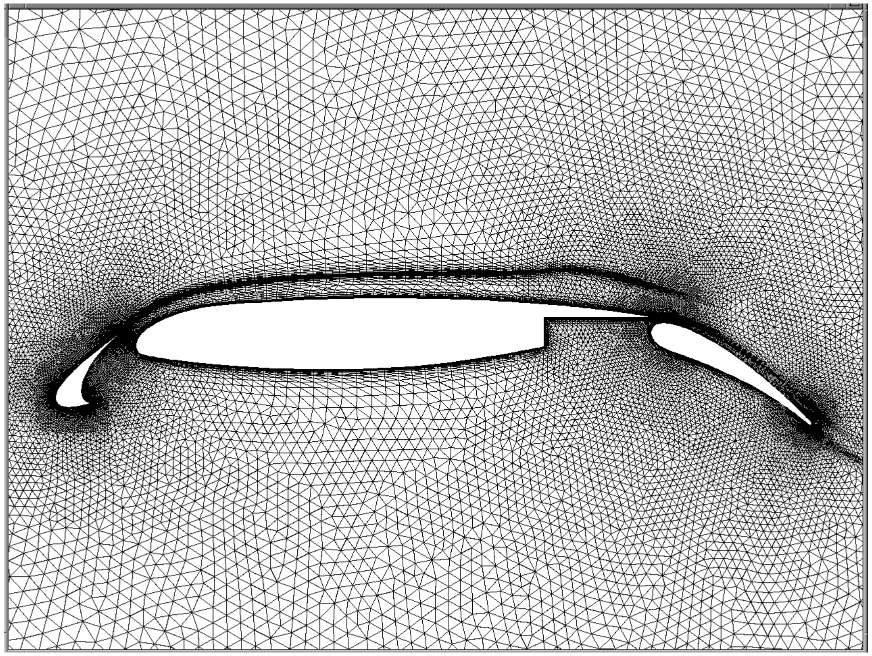
- *Newton-Krylov Provides Rapid Non-Linear Convergence*
- *Newton-Krylov Yields Modest Gains in Efficiency for Both Linear and Non-Linear Preconditioners*

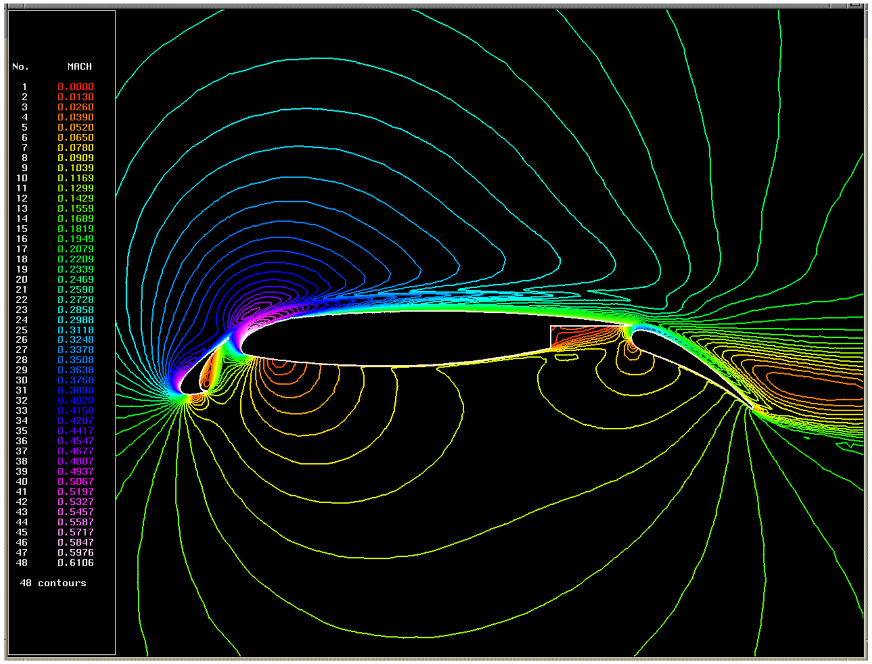
SOLUTION OF MULTI-ELEMENT AIRFOIL PROBLEM

- *Mixed Quad-Triangular Grid: 60,000 points*
- *1.E-06 Spacing at Wall*
- *Mach = 0.2, Incidence = 16 degrees, Reynolds Number = 9 million*
- *Among Most Difficult 2D Navier-Stokes Problems*

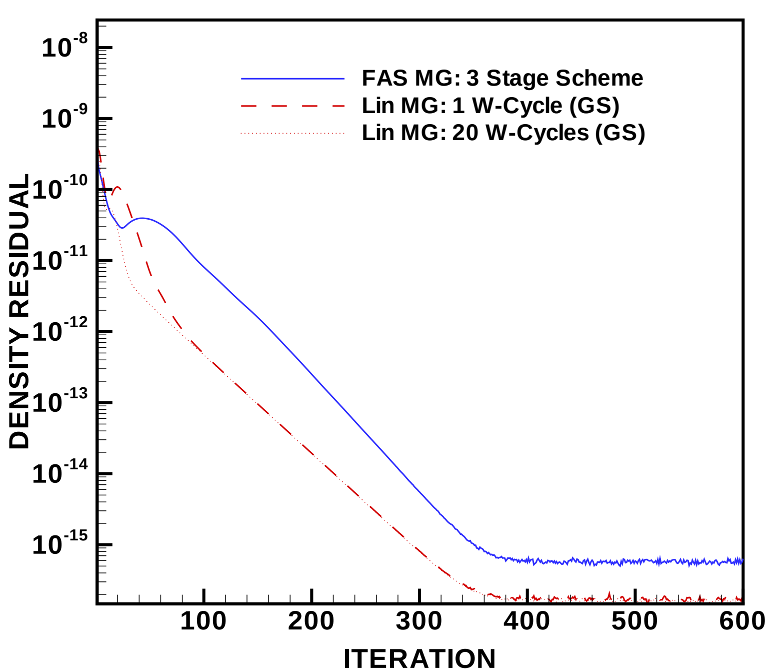
SOLUTION OF MULTI-ELEMENT AIRFOIL PROBLEM

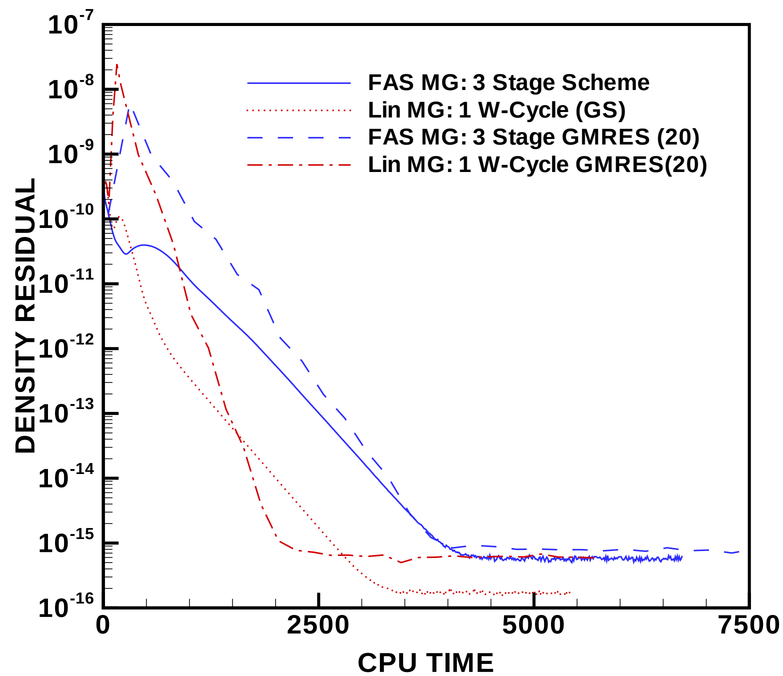




- *Mach = 0.2, Incidence = 16 degrees, Reynolds Number = 9 million*
- *Initial Conditions Pre-Converged 100 FAS Cycles*
- *Spalart-Allmaras Turbulence Model Frozen at Final Values*

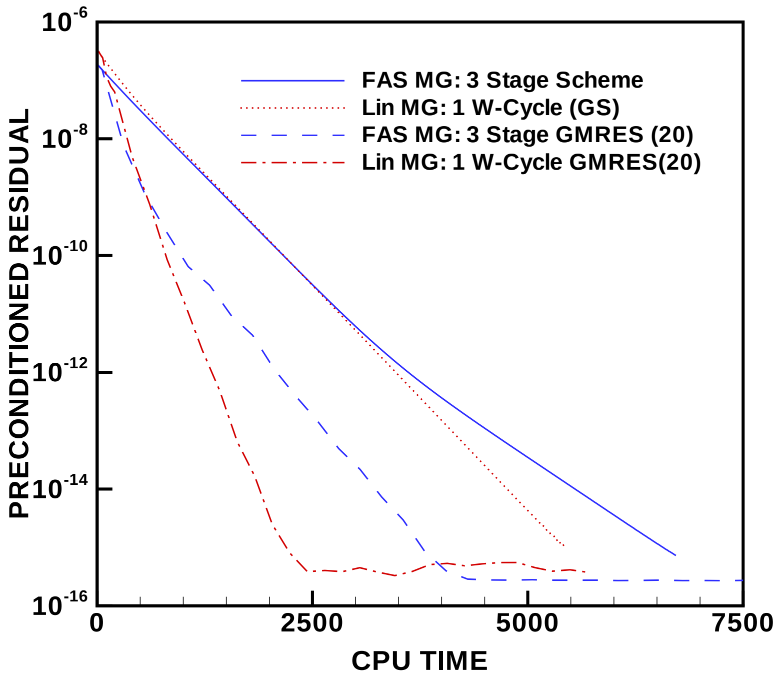
SOLUTION OF MULTI-ELEMENT AIRFOIL PROBLEM

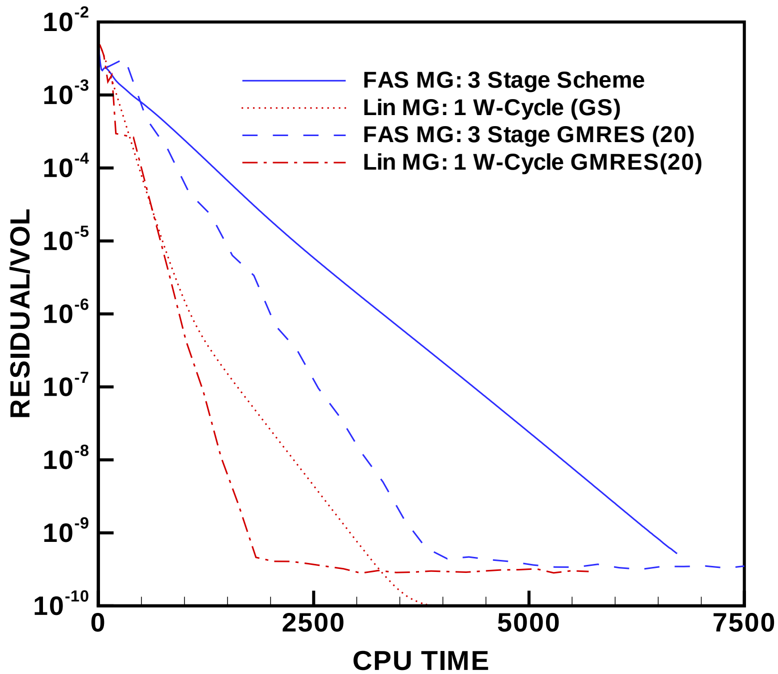




- *Much Slower Convergence for All Methods*
- *Convergence Limited by Inaccurate Jacobian*
- *FAS-Method Faster per Iteration than Linear Method Solving Linear System to Completion at Each Iteration*
- *Efficiency of Linear and Non-Linear Methods Comparable*
- *Newton-Krylov Provides Larger Speedups for Both Preconditioners*

SOLUTION OF MULTI-ELEMENT AIRFOIL PROBLEM





- *BEWARE: Definition of Convergence can Affect Conclusions*
- *Left Preconditioning*
- *Almost no Difference in Previous Cases*

CONCLUSIONS

- *For Exact (Newton) Jacobians*
  - *Linear and Non-Linear Methods Become Equivalent Asymptotically*
  - *Linear Methods often more efficient due to cheaper iteration cost*
- *For Inexact Jacobians*
  - *Efficiency determined by relative cost of linear versus non-linear iteration as well as efficiency of Defect-Correction Method*
  - *Effectiveness of Defect-Correction Method Appears to be Case Dependent*
  - *Newton-Krylov Effective when Poor Linearization Approximations Available*
  - *N-K Provides Similar Effectiveness using Lin. or Non-Lin. Preconditioners*
- *Only Asymptotic Convergence Considered*
- *Consider Parallel Scalability in Future*