PARTITION OF UNITY FOR THE STOKES PROBLEM ON NONMATCHING GRIDS

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Abstract. We consider the Stokes Problem on a plane polygonal domain $\Omega \subset \mathbb{R}^2$. We propose a finite element method for overlapping or nonmatching grids for the Stokes Problem based on the partition of unity method. We prove that the discrete inf-sup condition holds with a constant independent of the overlapping size of the subdomains. The results are valid for multiple subdomains and any spatial dimension.

1. Introduction

In the present literature the study of finite element method applied to overlapping grids is done mainly in the framework of mortar method or Lagrange multiplier (see [1, 12, 7]). A new finite element discretization for elliptic boundary value problems was introduced by Huang and Xu [10], using a partition of unity method which has the roots in [2]. A significant amount of literature was dedicated to numerical solutions of the Stokes problem (see e.g., [9, 5] and the references of this two books). By our knowledge not to much was done for solving discretization of the Stokes problem when overlapping grids or nonmatching grids are involved. In this paper, following the ideas of Huang and Xu, we shall introduce a conforming finite element method, using a partition of unity type argument for the steady-state Stokes problem.

2. The continuous Stokes problem problem and overlapping subdomains discretization

Even though the results hold in a more general context and for a general dimension, for clarity, we present the main ideas of the discretization method in case of two subdomains in $\mathbb{R}^2$. Given a bounded domain Let $\Omega \subset \mathbb{R}^2$, be a bounded domain with boundary $\partial\Omega$ and let Γ be a closed subset of $\partial\Omega$. By $H^1_0(\Omega;\Gamma)$ we denote the closure in H^1 topology of $C^\infty(\bar{\Omega})$ functions that vanish in a neighborhood of Γ.

The steady-state Stokes problem in the velocity-pressure formulation is :

Find the vector-valued function $\mathbf{u}$ and the scalar-valued function p satisfying

$$\begin{cases} -\Delta \mathbf{u} - \nabla p = \mathbf{F} & \text{in } \ \Omega, \\ \nabla \cdot \mathbf{u} = 0 & \text{in } \ \Omega, \\ \mathbf{u} = 0 & \text{on } \partial\Omega, \\ \int_\Omega p = 0 & . \end{cases} \tag{2.1}$$

Date: April 11, 2003.

Key words and phrases. non-matching grid, finite element method, partition of unity, Stokes problem.

This work is supported by the NSF Grant No. 0209497 (Multiscale Methods for Partial Differential equations).

Let $(\cdot,\cdot)_\Omega$, or simply $(\cdot,\cdot)$, denote the $L^2(\Omega)$-inner product applied to a pair of either scalar or vector functions. Similarly, let $\|\cdot\|_{0,\Omega}$, or simply $\|\cdot\|$ denote the $L^2(\Omega)$-norm. Define $\mathbf{V} = (H_0^1(\Omega))^2$ and $P = L_0^2(\Omega)$ the subspace of $L^2(\Omega)$ consisting of functions with zero mean value on Ω. The variational formulation of the problem (2.1) is

Find $(\mathbf{u}, p) \in (\mathbf{V}, P)$ such that

$$\begin{cases} a(\mathbf{u},\mathbf{v}) + b(\mathbf{v},p) = (\mathbf{F},\mathbf{v}) & \text{for all } \mathbf{v} \in \mathbf{V}, \\ b(\mathbf{u},q) = 0 & \text{for all } q \in P. \end{cases} \tag{2.2}$$

where a is the Dirichlet form on Ω defined by

$$a(\mathbf{u},\mathbf{v}) = \sum_{i=1}^{2} \int_\Omega \nabla u_i \cdot \nabla v_i \, dx.$$

and

$$b(\mathbf{v},q) = (q, \nabla\cdot\mathbf{v}).$$

We assume that the inf-sup condition

$$c_0\|p\| \le \sup_{\mathbf{v}\in\mathbf{V}} \frac{(p,\nabla\cdot\mathbf{v})}{\|\mathbf{v}\|_{1,\Omega}}, \quad \text{for all } p \in P, \tag{2.3}$$

holds for a positive constant c_0. Consequently, there is a unique solution $(\mathbf{u}, p) \in (\mathbf{V}, P)$ of (2.2). Let Ω be covered by a family of overlapping subdomains. For a better presentation of the main idea, we consider the case of two overlapping subdomains with polygonal shapes. Let Ω_1, Ω_2 be overlapping subdomains of Ω satisfying $\Omega = \Omega_1 \cup \Omega_2$ and $\Omega_0 = \Omega_1 \cap \Omega_2$, We further assume that Ω_1 and Ω_2 are partitioned by quassiuniform finite element triangulations $\mathcal{T}_1$ and $\mathcal{T}_2$ of maximal mesh sizes h_1 and h_2 (which might not match on Ω_0). We assume that Ω_0 is a strip-type domain of width $d = O(h_1)$ (the dotted region shown in Fig. 1).

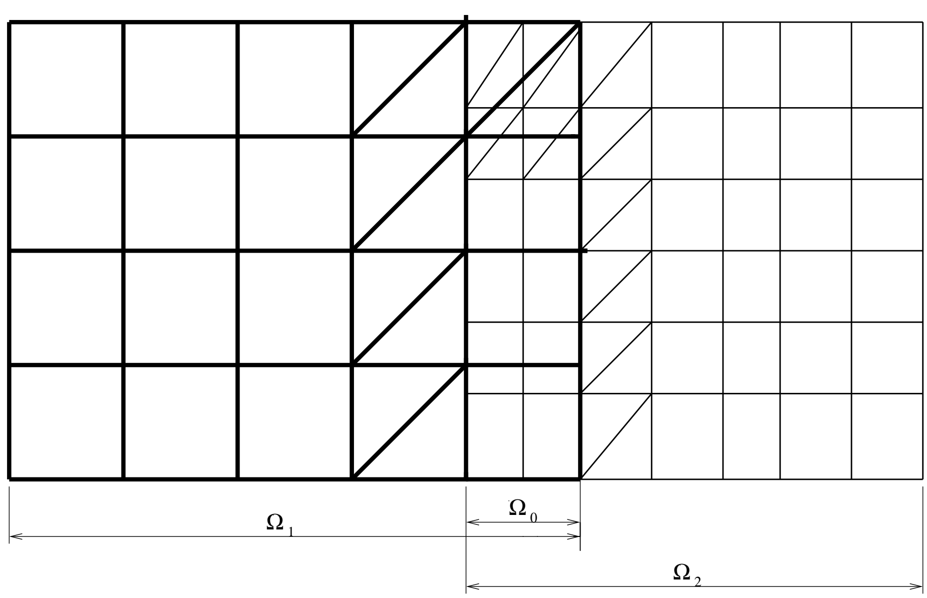

FIGURE 1. Overlapping grids

Next, we let $\{\phi_1, \phi_2\}$ be a partition of unity subordinate to the covering partition $\{\Omega_1, \Omega_2\}$ of Ω, i.e. $\phi_1 + \phi_2 = 1$, $0 \le \phi_i \le 1$, and $\|\nabla \phi_i\|_{\infty,\Omega} \le 1/d$. We further assume that $\phi_1 \equiv 1$ on $\Omega_1 \backslash \Omega_0$ and $\phi_1 \equiv 0$ on $\Omega_2 \backslash \Omega_0$.

To obtain a conforming discretization of the variational problem (2.2) we define first the following spaces

$$P_{h_i}(\Omega_i) := \{p \in C^0(\Omega_i) |\ p|_K \in \mathbb{P}_1\},$$

$$\hat{P}_{h_i}(\Omega_i) := \{p \in P_{h_i}(\Omega_i) |\ p = 0 \text{ on } \partial\Omega_i \backslash \partial\Omega\},$$

$$\mathbf{V}_{h_i}(\Omega_i) := \{\mathbf{v} \in (H_0^1(\Omega_i; \Omega \cap \partial\Omega_i))^2 |\ \mathbf{v}|_K \in \mathbb{P}_1\},$$

where, $\mathbb{P}_1$ denotes the set of polynomials in two variables of degree at most one. Using the above spaces, we are interested in building stable pairs $(\mathbf{V}_h, P_h)$, where $\mathbf{V}_h \subset \mathbf{V}$ and $P_h \subset P$, i.e., pairs $(\mathbf{V}_h, P_h)$ which satisfy the discrete inf-sup condition

$$c_0 \|p\| \le \sup_{\mathbf{v} \in \mathbf{V_h}} \frac{(p, \nabla \cdot \mathbf{v})}{\|\mathbf{v}\|_{1,\Omega}}, \quad \text{for all } p \in P_h. \tag{2.4}$$

If the above condition is satisfied then the discrete variational problem:

Find $(\mathbf{u}_h, p_h) \in \mathbf{V}_h \times P_h$ such that

$$\begin{cases} a(\mathbf{u}_h, \mathbf{v}) + b(\mathbf{v}, p_h) = (\mathbf{F}, \mathbf{v}) & \text{for all } \mathbf{v} \in \mathbf{V_h}, \\ b(\mathbf{u}_h, q) = 0 & \text{for all } q \in P_h, \end{cases} \tag{2.5}$$

has unique solution and the error satisfies,

$$|\mathbf{u} - \mathbf{u}_h|_{1,\Omega} + \|p - p_h\|_{0,\Omega} \le C(\inf_{\mathbf{v}_h \in \mathbf{V_h}} |\mathbf{u} - \mathbf{v}_h|_{1,\Omega} + \inf_{q_h \in P_h} \|p - q_h\|_{0,\Omega}),$$

with C depending on c_0, but independent of h (or the spaces $\mathbf{V_h}$ and P_h). In the next two sections we build stable pairs $(\mathbf{V_h}, P_h)$ which have also good approximation properties.

3. First mini-type stable pair

We introduce a space B of bubble functions associated with the "union " partition $\mathcal{T} := \mathcal{T}_1 \cup \mathcal{T}_2$ as follows. For a triangle T we define the bubble function B_T supported on T as the product of the nodal functions associated with the vertices of T. If $K = T_1 \cap T_2 \in \mathcal{T}_1 \cup \mathcal{T}_2$ we define

$$B_K := B_{T_1} \cdot B_{T_2}.$$

If $K = T_i$ for some $T_i \in \mathcal{T}_i$, $(i = 1, 2)$, then we just take $B_K := B_{T_i}$ (see Fig. 2). A composite, conforming finite element space for velocity can be defined as

$$\mathbf{V}_h \equiv \mathbf{V}_h(\Omega) := \phi_1 \mathbf{V}_{h_1} + \phi_2 \mathbf{V}_{h_2} + B^2.$$

The discrete pressure space we associate with $\mathbf{V}_h$ is

$$P_h := (\hat{P}_{h_1}(\Omega_1) + \hat{P}_{h_2}(\Omega_2)) \cap P.$$

Let $h := h_1 \ge h_2 = r h_1$, for some positive constant r. Before we state the main result of this section we introduce the following assumption:

- **(A1) There exists a positive constant c such that $|K| \cong ch^2$ for any $K \in \mathcal{T}$,**

where $|K|$ denotes the Lebesgue measure of $K \in \mathcal{T}$.

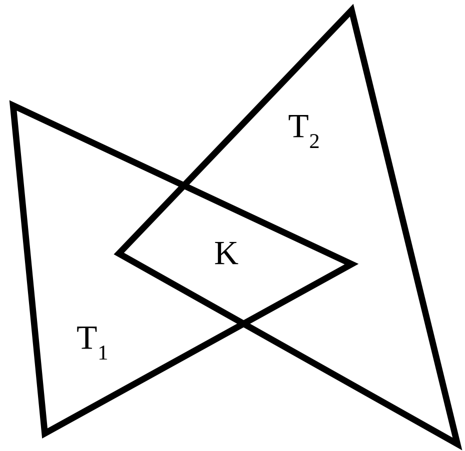

FIGURE 2. Overlapping triangles.

Theorem 3.1. *If* **(A1)** *is satisfied, then the pair* $(\mathbf{V}_h, P_h)$ *defined above is a stable pair.*

Proof. We will construct two operators $\Pi_1 : \mathbf{V} \to \mathbf{V}_h$, $\Pi_2 : \mathbf{V} \to \mathbf{V}_h$ with the following properties:

$$|\mathbf{v} - \Pi_1 \mathbf{v}|_{1,\Omega} \lesssim |\mathbf{v}|_{1,\Omega}, \quad \text{for all } \mathbf{v} \in \mathbf{V}, \tag{3.1}$$

$$|\Pi_2 (I - \Pi_1)\mathbf{v}|_{1,\Omega} \lesssim |\mathbf{v}|_{1,\Omega}, \quad \text{for all } \mathbf{v} \in \mathbf{V} \tag{3.2}$$

$$b(\mathbf{v} - \Pi_2 \mathbf{v}, q) = 0, \quad \text{for all } q \in P_h, \mathbf{v} \in \mathbf{V} \tag{3.3}$$

Having constructed Π_1 and Π_2, the operator $\Pi_h = \Pi_1 + \Pi_2(I - \Pi_1)$ satisfies the the hypothesis of Proposition II 2.8 in [5], for example, and the inf-sup condition follows according with this Proposition.

For $i = 1, 2$ let $\mathbf{V}_i := (H^1_0(\Omega_i; \Omega \cap \partial\Omega_i))^2$ and define $\Pi^i_1 : \mathbf{V}_i \to \mathbf{V}_{h_i}$ to be good regularization operators. For example, we can take Π^i_1 to be Clement-type operators . Thus,

$$\|\mathbf{v} - \Pi^i_1 \mathbf{v}\|_{0,\Omega_i} \lesssim h_i |\mathbf{v}|_{1,\Omega_i}, \quad \text{for all } \mathbf{v} \in \mathbf{V}_i, \tag{3.4}$$

and

$$|\mathbf{v} - \Pi^i_1 \mathbf{v}|_{1,\Omega_i} \lesssim |\mathbf{v}|_{1,\Omega_i}, \quad \text{for all } \mathbf{v} \in \mathbf{V}_i. \tag{3.5}$$

We define Π_1 as follows:

$$\Pi_1 \mathbf{v} := \phi_1 \Pi^1_1(\mathbf{v}_{|\Omega_1}) + \phi_2 \Pi^1_2(\mathbf{v}_{|\Omega_2})$$

Note that $\mathbf{v}_{|\Omega_i} \in \mathbf{V}_i$ and $\Pi_1 \mathbf{v} \in \mathbf{V}_h$. Thus Π_1 is well defined. In order to simplify the notation we denote $\Pi^1_i(\mathbf{v}_{|\Omega_i})$ simply by $\Pi^1_i \mathbf{v}$. Next, we verify that the operator Π_1 satisfies (3.1). We will prove first that

$$\|\mathbf{v} - \Pi_1 \mathbf{v}\|_{0,\Omega} \lesssim h |\mathbf{v}|_{1,\Omega}, \quad \text{for all } \mathbf{v} \in \mathbf{V}. \tag{3.6}$$

Indeed,

$$\|\mathbf{v}-\Pi_1\mathbf{v}\|_{0,\Omega} = \left\|\sum_{i=1}^{2}\phi_i(\mathbf{v}-\Pi_1^i\mathbf{v})\right\|_{0,\Omega} \le \left\|\sum_{i=1}^{2}\phi_i(\mathbf{v}-\Pi_1^i\mathbf{v})\right\|_{0,\Omega_i}$$
$$\le \sum_{i=1}^{2}\left\|\mathbf{v}-\Pi_1^i\mathbf{v}\right\|_{0,\Omega_i} \lesssim \sum_{i=1}^{2} h_i|\mathbf{v}|_{1,\Omega_i} \lesssim h|\mathbf{v}|_{1,\Omega}.$$

To justify (3.1) we have

$$|\mathbf{v}-\Pi_1\mathbf{v}|_{1,\Omega} \le \sum_{i=1}^{2}|\phi_i(\mathbf{v}-\Pi_1^i\mathbf{v})|_{1,\Omega_i} \le \sum_{i=1}^{2}|\nabla\phi_i(\mathbf{v}-\Pi_1^i\mathbf{v})|_{0,\Omega_i} + \sum_{i=1}^{2}|\mathbf{v}-\Pi_1^i\mathbf{v}|_{1,\Omega_i}$$
$$\lesssim d^{-1}\sum_{i=1}^{2}|\mathbf{v}-\Pi_1^i\mathbf{v}|_{0,\Omega_i} + \sum_{i=1}^{2}|\mathbf{v}|_{1,\Omega_i} \lesssim |\mathbf{v}|_{1,\Omega}.$$

Next, we define Π_2. For $\mathbf{v}\in\mathbf{V}$ and $K\in\mathcal{T}$,

$$\Pi_2\mathbf{v}_{|K} := \alpha B_K,$$

where $\alpha = \begin{pmatrix}\alpha_1\\ \alpha_2\end{pmatrix}$ is defined such that $\int_K(\mathbf{v}-\Pi_2\mathbf{v})\ dx = 0$ i.e.,

$$\alpha = \frac{\int_K \mathbf{v}\ dx}{\int_K B_K\ dx}$$

For $\mathbf{v}\in\mathbf{V}$ and $q\in P_h$ we have

$$b(\mathbf{v}-\Pi_2\mathbf{v},q) = -(\mathbf{v}-\Pi_2\mathbf{v},\nabla q) = -\sum_{K\in\mathcal{T}}\nabla q\int_K(\mathbf{v}-\Pi_2\mathbf{v})\ dx = 0.$$

Thus (3.6) holds and all we have to verify is that (3.2) holds. Let us note that

$$|\Pi_2\mathbf{v}|_{1,K} \lesssim h^{-1}|\mathbf{v}|_{0,K} \qquad \text{for all } \mathbf{v}\in\mathbf{V}. \tag{3.7}$$

The proof of (3.7) is a consequence of the following two estimates.

$$|\Pi_2\mathbf{v}|_{1,K}^2 = (\alpha_1^2+\alpha_2^2)\int_K|\nabla B_K|^2 \lesssim |\alpha|^2|K|h^{-2},$$

and

$$|\alpha|^2 = \frac{|\int_K \mathbf{v}\ dx|^2}{|\int_K B_K\ dx|^2} \lesssim \frac{|K||\mathbf{v}|_{0,K}^2}{h^2}$$

Thus, from (3.7) and (3.6) we obtain

$$|\Pi_2(I-\Pi_1)\mathbf{v}|_{1,\Omega}^2 = \sum_{K\in\mathcal{T}}|\Pi_2(I-\Pi_1)\mathbf{v}|_{1,K}^2 \lesssim \sum_{K\in\mathcal{T}}h^{-2}|(I-\Pi_1)\mathbf{v}|_{0,K}^2 \lesssim |\mathbf{v}|_{1,\Omega}^2,$$

which proves that (3.2) holds and concludes the the proof of the theorem. □

Remark 3.1. *In the special case when any $K\in\mathcal{T}$ is a triangle in either T_1 or T_2 (any $T_1\in\mathcal{T}_1$, $T_1\subset\Omega_0$ is an union of triangles of $\mathcal{T}_2$ and any $K\in\mathcal{T}$ which is not subset of Ω_0 belongs to either T_1 or T_2), we have that* **(A1)** *is satisfied. Moreover we have*

- **(A1)' There exists a positive constant c such that $|T_i|\cong ch_i^2$ for any $K=T_i\in\mathcal{T}$.**

Following the proof of the above theorem in this particular case, we deduce that the constants which are involved in (3.1) *and* (3.2) *are also independent of the ratio* $r = h_2/h_1$. *Consequently, the inf-sup condition holds with a constant independent of* h_2, h_1 *and* r.

Remark 3.2. *According with* [10] *the space* $\mathbf{V}_h$ *has good approximation properties,*

$$\inf_{\mathbf{v}_h \in \mathbf{V}_h} \|\mathbf{v} - \mathbf{v}_h\|_{1,\Omega} \lesssim h_1 \|v\|_{1,\Omega_1} + h_2 \|v\|_{1,\Omega_2}, \quad \text{for all } \mathbf{v} \in \mathbf{V}.$$

If $\hat{P}_{h_1}(\Omega_1) + \hat{P}_{h_2}(\Omega_2)$ *is a linear space which contains the constant function, then one can prove that the space* P_h *has the following approximation property*

$$\inf_{p_h \in P_h} \|P - P_h\|_{0,\Omega} \lesssim h \|P\|_{1,\Omega}, \quad \text{for all } p_h \in P_h.$$

Therefore the pair $(\mathbf{V}_h, P_h)$ *has good approximation properties and is a stable pair.*
On the other hand, if $\hat{P}_{h_1}(\Omega_1) + \hat{P}_{h_2}(\Omega_2)$ *is a linear space which does not have good approximation properties we can consider for the discrete pressure space* P_h *a partition of unity type space and modify accordingly the velocity space. This is the subject of the next section.*

4. Second mini-type stable pair

A discrete pressure space P_h with good approximation properties (see [10]) is the space

$$P_h := (\phi_1 P_{h_1}(\Omega_1) + \phi_2 P_{h_2}(\Omega_2)) \cap P.$$

Since the pressure space is enriched (on the overlapping region), in order to have satisfied the inf-sup condition, we have to enrich the velocity space also. As in the previous section we define a bubble space B. For each $K \in \mathcal{T}$, $K \subset \Omega_0$ we let $B_K^j, j = 1, 2$ to be two bubble functions supported on$K$ which have certain properties and are to be specified later. For each one the of remaining regions $K \in \mathcal{T}$ we consider only one bubble function defined as in the first case. We let B to be the span of all this bubble functions and define the discrete space $\mathbf{V}_h$ as before

$$\mathbf{V}_h := \phi_1 \mathbf{V}_{h_1} + \phi_2 \mathbf{V}_{h_2} + B^2.$$

Theorem 4.1. *If* **(A1)** *is satisfied, then the new pair* $(\mathbf{V}_h, P_h)$ *defined above is a stable pair.*

Proof. We follow the construction procedure revealed in Theorem 3.1. The Π_1 operator is the one defined in the proof of Theorem 3.1. Next, we define Π_2 such that (3.2) and (3.6) are satisfied. Let $\phi_1 := \phi$ and $\phi_2 := 1 - \phi$. To simplify the computation we will assume that ϕ is a linear function in only one variable, say x. Thus, for any $q \in P_h$ and any $K \in \mathcal{T}$, $K \subset \Omega_0$ we have that

$$\nabla q_{|K} \in \operatorname{span} \left\{ \begin{pmatrix} x \\ 0 \end{pmatrix}, \begin{pmatrix} y \\ x \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$$

We define Π_2 as follows

$$\Pi_2 \mathbf{v}_{|K} := \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} B_K^1 + \begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix} B_K^2,$$

if $K \in \mathcal{T}$, $K \subset \Omega_0$ and $\Pi_2 \mathbf{v}_{|_K} := \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} B_K$,, if $K \in \mathcal{T}$ and $K \subset \Omega_i \backslash \Omega_0$, $i = 1, 2$. The constants $\alpha = \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix}$ and $\beta = \begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix}$ are determined such that $\int_K (\mathbf{v} - \Pi_2 \mathbf{v})\ dx = 0$. Here, in the second case, B_K is the bubble function defined in the previous section. The justification of (3.2) is similar and we only need to prove that (3.7) holds. For $K \in \mathcal{T}$ and $K \subset \Omega_i \backslash \Omega_0$, $i = 1, 2$ the proof was done in the previous section. We will focus now on the case $K \in \mathcal{T}$, $K \subset \Omega_0$. From the definition of Π_2 and the condition $\int_K (\mathbf{v} - \Pi_2 \mathbf{v})\ dx = 0$ we deduce that

$$
\begin{cases}
\alpha_1 (B_K^1, x) + \beta_1 (B_K^2, x) & = (v_1, x) \\
\alpha_1 (B_K^1, y) + \beta_1 (B_K^2, y) + \alpha_2 (B_K^1, x) + \beta_2 (B_K^2, x) & = (v_1, y) + (v_2, x) \\
\alpha_1 (B_K^1, 1) + \beta_1 (B_K^2, 1) & = (v_1, 1) \\
\alpha_2 (B_K^1, 1) + \beta_2 (B_K^2, 1) & = (v_2, 1).
\end{cases} \tag{4.1}
$$

The system has unique solution if and only if

$$
det_K := det \begin{pmatrix} (B_K^1, x), (B_K^2, x) \\ (B_K^1, 1), (B_K^2, 1) \end{pmatrix} \neq 0. \tag{4.2}
$$

Let us assume that B_K^1 and B_K^2 are chosen such that (4.2) is satisfied. Then one can solve for α and β. For example we have

$$
\alpha_1 = \frac{1}{det_K} \left((B_K^2, 1)(v_1, x) - (B_K^2, x)(v_1, 1) \right). \tag{4.3}
$$

If we further assume that

$$
|\nabla B_K^i| \lesssim h^{-2} \text{ on } K. \tag{4.4}
$$

Then, using **(A1)**, we get

$$
|\Pi_2 \mathbf{v}|_{1,K}^2 = (\alpha_1^2 + \alpha_2^2) \int_K |\nabla B_K^1|^2 + (\beta_1^2 + \beta_2^2) \int_K |\nabla B_K^2|^2 \lesssim (|\alpha|^2 + |\beta|^2). \tag{4.5}
$$

On the other hand from (4.3) we have

$$
|\alpha_1| \leq \frac{1}{|det_K|} h^2 |v_1|_{0,K} h.
$$

If B_K^1 and B_K^2 are chosen such that we have

$$
det_K \gtrsim h^4, \quad \text{for all } K \in \mathcal{T},\ K \subset \Omega_0, \tag{4.6}
$$

then

$$
|\alpha_1| \lesssim h^{-1} |\mathbf{v}|_{0,K},
$$

and similar estimates hold for α_2, β_1 and β_2. Hence, via (4.5), we have that (3.6) holds and consequently (3.7) is satisfied. All we have left is to prove that we can choose B_K^1 and B_K^2 such that (4.4) and (4.6) are satisfied. One way of choosing B_K^1 and B_K^2 is to take $B_K^1 = B_K$, where B_K was defined in the previous section, and then define $B_K^2 := B_K(\gamma_K - \phi)$, where γ_K is chosen such that $(B_K^2, 1) = 0$,

$$
\gamma_K := \frac{\int_k B_K \phi\ dx}{\int_k B_K\ dx}.
$$

□

5. Nonoverlapping Nonmatching grids

Let Ω be split in two nonoverlapping subdomains Ω_1 and Ω_2 and let Γ be the interface between Ω_1 and Ω_2. As in the overlapping case we assume that Ω_1 and Ω_2 are partitioned by quassiuniform finite element triangulations $\mathcal{T}_1$ and $\mathcal{T}_2$ of maximal mesh sizes h_1 and h_2 ($h_1 \geq h_2$). The grid on the interface Γ does not match the two partitions. We extend the mesh of Ω_1 inside Ω_2 so that the overlapping meshed region is a strip of size $d = O(h_1)$. In this way we have reduced the setting to the case of overlapping subdomains.(see Fig. 3).

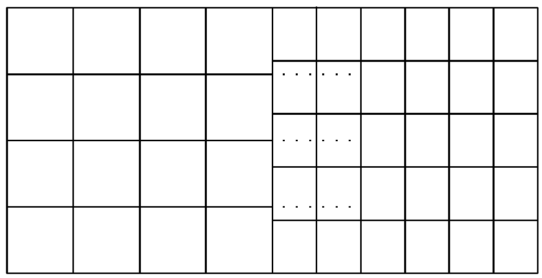

Figure 3. Nonoverlapping grids and simple extension to overlapping.

6. Conclusions

- The method can be extended with no difficulties to the more subdomains case or the multidimensional case.
- At the theoretical level it is simpler than the mortar method.
- If the discrete approximation spaces are spaces of continuous piecewise linear functions then the partition of unity functions could be chosen to be piecewise linear functions also.
- The condition **(A1)** is too restrictive. In practice, we can slightly change the mesh by moving points of the mesh towards other very close points or edges.
- In the nonoverlapping case we can extend the mesh of one domain as "submesh" of a neighboring subdomain.
- We conjecture that other classical stable pairs for subdomains. For example $\mathbb{P}_2 - \mathbb{P}_1$ "subspaces" could be glued by partition of unity method in order to construct stable pairs with good approximation properties.
- Adding more bubble functions for the overlapping regions in the velocity space would improve the stability.
- The partition of unity method could be similarly involved for mixed finite element methods on nonmatching grids.

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