BLACK-BOX PRECONDITIONING FOR MIXED FORMULATION OF SECOND-ORDER ELLIPTIC PROBLEMS

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Abstract. Raviart-Thomas mixed finite element approximation to second-order elliptic PDEs with variable diffusion coefficients is well understood. The resulting symmetric and indefinite linear system is ill-conditioned with repect to the discretisation parameter and the PDE coefficients. Preconditioning strategies commonly focus on reduced, denser symmetric positive definite systems and require nested iteration. This defficiency is avoided if preconditioned MINRES is applied to the full indefinite system. A practical preconditioning scheme is proposed, the key components of which are diagonal scaling for a weighted mass matrix and a fast solver for a scalar diffusion operator based on black-box algebraic multigrid (AMG). Eigenvalue bounds are derived for the preconditioned system matrix. Numerical results are presented to illustrate that the preconditioner is optimal with respect to the discretisation parameter and is robust with respect to the PDE coefficients.

Key words: saddle-point problems, variable coefficients, mixed finite elements, Raviart-Thomas, MINRES, AMG, preconditioning.

1. Introduction. Let Ω be a bounded domain in $\mathbb{R}^2$. We consider scalar, second-order elliptic problems of the form,

$$\begin{aligned} -\nabla \cdot \mathcal{A}\nabla p &= f \quad \text{in } \Omega, \\ p &= g \quad \text{on } \partial\Omega_D, \\ \mathcal{A}\nabla p \cdot \vec{n} &= 0 \quad \text{on } \partial\Omega_N. \end{aligned} \tag{1.1}$$

where $\partial\Omega_D \neq \emptyset$, $\vec{n}$ denotes the unit outward normal vector and f and g are given functions. $\mathcal{A} = \sigma(\vec{x})\mathcal{I}$ is a bounded, symmetric and uniformly positive definite matrix-valued function with,

$$0 < \sigma_{min} \leq \sigma(\vec{x}) \leq \sigma_{max} < \infty, \quad \forall \vec{x} \in \Omega. \tag{1.2}$$

Boundary-value problems of this type occur in mathematical models of important physical processes such as fluid flow in porous media. To fix ideas, we call p and $\vec{u} = \mathcal{A}\nabla p$ the pressure and velocity solutions respectively. Mixed finite element methods are favoured when $\vec{u}$ is the variable of interest and rough coefficients are present since post-processing primal pressure solutions leads to loss of accuracy. Moreover, mixed methods conserve mass locally, a crucial feature in the modelling of groundwater flow.

Discretising (1.1) using the Raviart-Thomas finite element spaces (see [7]) leads to a saddle-point system of the form,

$$\begin{pmatrix} A & B^T \\ B & 0 \end{pmatrix} \begin{pmatrix} \underline{u} \\ \underline{p} \end{pmatrix} = \begin{pmatrix} \underline{g} \\ \underline{f} \end{pmatrix}. \tag{1.3}$$

We are interested in robust, parameter-free solution schemes for (1.3). The matrix $A \in \mathbb{R}^{n \times n}$ is symmetric and positive definite and the discretisation is inf-sup stable (see [2]) so B has full rank. Without preconditioning, Krylov solution methods are not robust with respect to the discretisation parameter or the PDE coefficients; preconditioners are essential. An *optimal* preconditioner is a matrix operator that accelerates the convergence

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rate of the chosen iterative solver so that convergence to a fixed tolerance is independent of the problem parameters.

Two distinct types of preconditioning scheme are possible. The first consists of solving the denser positive definite Schur complement system,

$$S\underline{p} \;=\; BA^{-1}\underline{g} - \underline{f}, \tag{1.4}$$

with $S = BA^{-1}B^T$, using the preconditioned conjugate gradient (CG) method. This requires matrix-vector multiplication with S and thus computation of the action of A^{-1}. Here, the condition number of A depends on the PDE coefficients and so A also requires a preconditioner. The result is nested iteration, the success of which requires fine-tuning inner and outer stopping tolerances or other parameters (see [10]). Alternatively, system (1.3) can be solved directly using the minimal residual method (MINRES). It is easy to see that an optimal preconditioner for this is,

$$P = \begin{pmatrix} A & 0 \\ 0 & S \end{pmatrix}, \tag{1.5}$$

but clearly, an exact implementation is infeasible. However, S corresponds to a discrete scalar diffusion operator for which fast multigrid solvers are available. For positive definite systems, it is well known that simple multigrid methods accelerated by CG can be more effective than sophisticated multigrid algorithms used as solvers (see [12]) even for problems with complicated coefficients. We will show that applying basic Ruge and Stüben AMG as a preconditioner for S and combining this with diagonal scaling for A, yields an optimal and robust preconditioner for the indefinite system (1.3).

First, it is instructive to recall convergence properties of MINRES. Let us denote the symmetric and indefinite system (1.3) by $C\underline{x} = \underline{b}$ and suppose that a preconditioner with symmetric and positive definite blocks,

$$P = \begin{pmatrix} P_1 & 0 \\ 0 & P_2 \end{pmatrix}, \tag{1.6}$$

has been chosen. Given a zero initial guess $\underline{x}_0$, the iterative solver MINRES (see [4]) applied to the symmetrically preconditioned system,

$$\begin{pmatrix} P_1^{-\frac{1}{2}}AP_1^{-\frac{1}{2}} & P_1^{-\frac{1}{2}}B^TP_2^{-\frac{1}{2}} \\ P_2^{-\frac{1}{2}}BP_1^{-\frac{1}{2}} & 0 \end{pmatrix}\begin{pmatrix} \underline{y} \\ \underline{z} \end{pmatrix} = \begin{pmatrix} P_1^{-\frac{1}{2}}\underline{g} \\ P_2^{-\frac{1}{2}}\underline{f} \end{pmatrix}, \quad \begin{pmatrix} \underline{y} \\ \underline{z} \end{pmatrix} = \begin{pmatrix} P_1^{\frac{1}{2}}\underline{u} \\ P_2^{\frac{1}{2}}\underline{p} \end{pmatrix}, \tag{1.7}$$

generates a sequence of iterates $\underline{x}_k$, belonging to the Krylov space,

$$K_k \;=\; \underline{x}_0 + \mathtt{span}\left(P^{-1}\underline{r}_0, P^{-1}CP^{-1}\underline{r}_0, \ldots, \left(P^{-1}C\right)^{k-1}P^{-1}\underline{r}_0\right),$$

with the minimisation property,

$$\| \underline{b} - C\underline{x}_k \|_{P^{-1}} = \min_{\underline{x}\in K_k} \| \underline{b} - C\underline{x} \|_{P^{-1}},$$

where $\underline{r}_k \;=\; \underline{b} - C\underline{x}_k$ is the kth residual and $\| \underline{v} \|_{P^{-1}}^2 = \underline{v}^T P^{-1}\underline{v}$.

Since P is symmetric, it can be shown that a sharp upper bound for the residual error after k iterations (see [3, p.51]) is given by,

$$\frac{\| \underline{r}^{(k)} \|_{P^{-1}}}{\| \underline{r}^{(0)} \|_{P^{-1}}} \leq \min_{p_k} \max_{i=1:(n+m)} | \, p_k(\lambda_i) \, |, \tag{1.8}$$

where p_k is a polynomial of degree k satisfying $p_k(0) = 1$, and $\{\lambda_i\}_{i=1}^{n+m}$ denotes the eigenvalues of $P^{-1}C$. Thus, the convergence rate of MINRES is completely determined by the eigenvalue spectrum of $P^{-1}C$. For a robust solution scheme, we seek P such that inclusion intervals for the values $\{\lambda_i\}_{i=1}^{n+m}$ are independent of all the problem parameters.

1.1. Notation. $\Omega \subseteq I\!R^2$ is a polygon with boundary $\partial\Omega$. $L^2(\Omega)$ is the usual space of square-integrable scalar functions with inner-product $(\cdot,\cdot)$. For the space of vector functions $L^2(\Omega)^2$, the definition is understood to hold componentwise. We write $\|\cdot\|_0$ to denote the Lebesgue measure for both spaces. The subspace,

$$H(div;\Omega) = \left\{\vec{v} \in L^2(\Omega)^2 \mid \nabla\cdot\vec{v} \in L^2(\Omega)\right\},$$

contains vectors with square-integrable divergence. The associated inner-product is $(\vec{u},\vec{v})_{div} = (\vec{u},\vec{v}) + (\nabla\cdot\vec{u},\nabla\cdot\vec{v})$ and we denote the induced norm $\|\cdot\|_{div}$.

2. Raviart-Thomas formulation. Substituting $\vec{u} = \mathcal{A}\nabla p$ in (1.1) yields the first-order system,

$$\begin{array}{rcll} \mathcal{A}^{-1}\vec{u} - \nabla p &=& 0, & \\ \nabla\cdot\vec{u} &=& -f & \text{in } \Omega, \\ p &=& g & \text{on } \partial\Omega_D, \\ \vec{u}\cdot\vec{n} &=& 0 & \text{on } \partial\Omega_N. \end{array} \tag{2.1}$$

To formulate the weak problem, we choose finite-dimensional subspaces, $W_h \subset L^2(\Omega)$ and $\vec{V}_h \subset H_{0,N}(div;\Omega) = \{\vec{v} \in H(div;\Omega) \mid \vec{v}\cdot\vec{n} = 0 \text{ on } \partial\Omega_N\}$ consisting of piecewise polynomials defined on a partition of Ω into uniform triangles of edge length h. Then we look for $\vec{u}_h \in \vec{V}_h$, $p_h \in W_h$ satisfying,

$$\begin{array}{rcll} (\mathcal{A}^{-1}\vec{u}_h,\vec{v}) + (p_h,\nabla\cdot\vec{v}) &=& < g,\vec{v}\cdot\vec{n} > & \forall\vec{v}\in\vec{V}_h, \\ (w,\nabla\cdot\vec{u}_h) &=& -(f,w) & \forall w\in W_h, \end{array} \tag{2.2}$$

where $\langle g,\vec{v}\cdot\vec{n}\rangle = \int_{\partial\Omega_D} g\,\vec{v}\cdot\vec{n}\,ds$.

We choose the lowest-order Raviart-Thomas finite element spaces. Stability (see [2]) of the method is established in [7] in the norms $\|\cdot\|_{div}$ and $\|\cdot\|_0$. For a given triangulation T_h of Ω we have,

$$\vec{V}_h = \{\vec{v} \in H(div;\Omega) \mid \vec{v}\,|_T \in (P_0^2(T) + [x,y]P_0(T)) \quad \forall T \in T_h\} \tag{2.3}$$

and $W_h = \{w \in L^2(\Omega) \mid w\ |_T \in P_0(T) \quad \forall T \in T_h\}$ where $P_0(T)$ denotes the set of polynomials of degree zero defined on triangle T. Velocities are linear in each component; the pressure space consists of piecewise constant functions. For an $H(div;\Omega)$ conforming velocity approximation, the degrees of freedom are normal components at edge midsides; the pressure solution is sampled at element centroids (see Fig. 2.1).

Choosing bases $\vec{V}_h = \mathtt{span}\left\{\vec{\psi}_i\right\}_{i=1}^{n}$ and $W_h = \mathtt{span}\left\{\phi_j\right\}_{j=1}^{m}$, we obtain precisely the linear algebra problem (1.3), with matrices A and B defined by,

$$\begin{aligned} A(i,j) &= \left(\mathcal{A}^{-1}\vec{\varphi}_i,\vec{\varphi}_j\right) \quad i,j = 1:n, \\ B(k,j) &= (\nabla\cdot\vec{\varphi}_j,\phi_k) \quad k = 1:m,\ j = 1:n, \end{aligned}$$

and vectors $g(i) = \langle g,\vec{\varphi}_i\cdot\vec{n}\rangle$, $f(k) = -(f,\phi_k)$ for $i = 1:n$ and $k = 1:m$. For variable diffusion coefficients, the integration is performed by sampling $\mathcal{A}$ at element centroids.

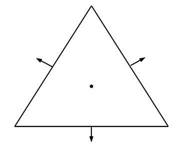

FIG. 2.1. *The lowest order triangular Raviart-Thomas element*

3. Preconditioning Strategy. Preconditioning strategies for the indefinite problem (1.3) are proposed in [10], [1],[5] and [13]. The first work suggests the preconditioner,

$$P = \begin{pmatrix} I & 0 \\ 0 & LL^T \end{pmatrix}, \tag{3.1}$$

where L is the incomplete Cholesky factor of BB^T. Optimal convergence can only be achieved with this scheme for very benign coefficients and by putting substantial effort into selecting fill-in and drop tolerance parameters. The second approach incorporates a multigrid preconditioner for an $H(div)$ operator for the case of constant diagonal coefficients. Black-box multigrid algorithms for elliptic problems are not applicable in that formulation. An exact preconditioner incorporating a weighted $H(div)$ operator is considered in [5]. In [13], an exact preconditioner for a modified saddle-point problem is discussed. The success of that scheme depends on the choice of two acceleration parameters.

Following [5], we propose the block-diagonal preconditioner,

$$P = \begin{pmatrix} D_A & 0 \\ 0 & P_S \end{pmatrix}, \tag{3.2}$$

where $D_A = \texttt{diag}(A)$ and P_S is 1 V-cycle of black-box AMG applied to the approximate Schur-complement $S_D = BD_A^{-1}B^T$. Our choice is motivated by the observation that for coefficients $\mathcal{A}$ of the form,

$$\mathcal{A} = \begin{pmatrix} \sigma(\vec{x}) & 0 \\ 0 & \sigma(\vec{x}) \end{pmatrix}, \tag{3.3}$$

diagonal scaling for A is an optimal choice, independent of $\sigma(\vec{x})$ (see Lemma 3.4 below). Moreover, the Schur-complement represents the generalised diffusion operator $\nabla \cdot \mathcal{A} \nabla$ and AMG is known to be an efficient *solver* for linear systems arising from discretisations of such operators (see [8]).

We now prove that the preconditioner (3.2) is optimal. Our starting point is the eigenvalue bound established by Rusten and Winther in [10].

THEOREM 3.1. *Let $0 < \mu_1 \ldots \leq \mu_n$ be the eigenvalues of A and let $0 < \rho_1 \ldots \leq \rho_m$ be the singular values of B, then the eigenvalues of the indefinite system matrix (1.3) lie in the union of the intervals,*

$$\begin{gathered} \left[\frac{1}{2}\left(\mu_1 - \sqrt{\mu_1^2 + 4\rho_m^2}\right), \frac{1}{2}\left(\mu_n - \sqrt{\mu_n^2 + 4\rho_1^2}\right)\right] \\ \cup \left[\mu_1, \frac{1}{2}\left(\mu_n + \sqrt{\mu_n^2 + 4\rho_m^2}\right)\right]. \end{gathered} \tag{3.4}$$

Proof. See [10]. □

Consider symmetric preconditioning for (1.3) with the exact (no AMG) version of the suggested preconditioner. We obtain,

$$P^{-\frac{1}{2}}CP^{-\frac{1}{2}} = \begin{pmatrix} D_A^{-\frac{1}{2}}AD_A^{-\frac{1}{2}} & D_A^{-\frac{1}{2}}B^T\left(BD_A^{-1}B^T\right)^{-\frac{1}{2}} \\ \left(BD_A^{-1}B^T\right)^{-\frac{1}{2}}BD_A^{-\frac{1}{2}} & 0 \end{pmatrix} = \begin{pmatrix} \tilde{A} & \tilde{B}^T \\ \tilde{B} & 0 \end{pmatrix}.$$

Applying Theorem 3.1 to this preconditioned saddle-point system leads, trivially, to the following result.

COROLLARY 3.2. *Let* $0 < \tilde{\mu}_1 \ldots \leq \tilde{\mu}_n$ *be the eigenvalues of* $D_A^{-1}A$*, then the eigenvalues of,*

$$P^{-\frac{1}{2}}\begin{pmatrix} A & B^T \\ B & 0 \end{pmatrix}P^{-\frac{1}{2}}, \ \textit{with} \ P = \begin{pmatrix} D_A & 0 \\ 0 & BD_A^{-1}B^T \end{pmatrix},$$

lie in the union of the intervals,

$$\left[\frac{1}{2}\left(\tilde{\mu}_1 - \sqrt{\tilde{\mu}_1^2+4}\right), \frac{1}{2}\left(\tilde{\mu}_n - \sqrt{\tilde{\mu}_n^2+4}\right)\right] \cup \left[\tilde{\mu}_1, \frac{1}{2}\left(\tilde{\mu}_n + \sqrt{\tilde{\mu}_n^2+4}\right)\right].$$

Proof. Observe that

$$\tilde{B}\tilde{B}^T = \left(BD_A^{-1}B^T\right)^{-\frac{1}{2}}BD_A^{-\frac{1}{2}}D_A^{-\frac{1}{2}}B^T\left(BD_A^{-1}B^T\right)^{-\frac{1}{2}} = I,$$

where I is the identity matrix. The result follows immediately from Theorem 3.1 since the singular values of $\tilde{B}$ satisfy $\tilde{\rho}_1 = \ldots = \tilde{\rho}_m = 1$ and $\tilde{A}$ has the same eigenvalue spectrum as $D_A^{-1}A$. ☐

Now, given *any* preconditioner P_S for $BD_A^{-1}B^T$ satisfying,

$$\theta^2 \leq \frac{\underline{p}^t BD_A^{-1}B^T\underline{p}}{\underline{p}^t P_S\underline{p}} \leq \Theta^2 \quad \forall \underline{p} \in I\!R^m \setminus \{\underline{0}\}.$$

for some constants θ and Θ, Theorem 3.1 can be extended to obtain the following theoretical eigenvalue bound for the indefinite preconditioned system.

COROLLARY 3.3. *Let* $0 < \tilde{\mu}_1 \ldots \leq \tilde{\mu}_n$ *be the eigenvalues of* $D_A^{-1}A$*, then the eigenvalues of,*

$$P^{-\frac{1}{2}}\begin{pmatrix} A & B^T \\ B & 0 \end{pmatrix}P^{-\frac{1}{2}}, \ \textit{with} \ P = \begin{pmatrix} D_A & 0 \\ 0 & P_S \end{pmatrix},$$

lie in the union of the intervals,

$$\left[\frac{1}{2}\left(\tilde{\mu}_1 - \sqrt{\tilde{\mu}_1^2+4\Theta^2}\right), \frac{1}{2}\left(\tilde{\mu}_n - \sqrt{\tilde{\mu}_n^2+4\theta^2}\right)\right] \cup \left[\tilde{\mu}_1, \frac{1}{2}\left(\tilde{\mu}_n + \sqrt{\tilde{\mu}_n^2+4\Theta^2}\right)\right].$$

Proof. The result follows from Theorem 3.1 and Corollary 3.2 with $\tilde{B} = P_S^{-\frac{1}{2}}BD_A^{-\frac{1}{2}}$. Observe that,

$$\begin{aligned} \underline{p}^T\tilde{B}\tilde{B}^T\underline{p} &= \underline{p}^T P_S^{-\frac{1}{2}}BD_A^{-1}B^T P_S^{-\frac{1}{2}}\underline{p} \\ &\leq \Theta^2\underline{p}^T P_S^{-\frac{1}{2}}P_S P_S^{-\frac{1}{2}}\underline{p} \\ &= \Theta^2\underline{p}^T\underline{p}. \end{aligned}$$

That is, the maximum singular value $\tilde{\rho}_m$ of $\tilde{B}$ satisfies, $\tilde{\rho}_m^2 \leq \Theta^2$. Similarly, it can be shown that $\tilde{\rho}_1^2 \geq \theta^2$. □

In section 4 we show that black-box AMG is an optimal choice for P_S yielding constants θ and Θ that are independent of h. By Corollary 3.3, an optimal eigenvalue bound is thus obtained if and only if, the constants $\tilde{\mu}_1$ and $\tilde{\mu}_n$ satisfy,

$$\tilde{\mu}_1 \leq \frac{\underline{u}^t A \underline{u}}{\underline{u}^t D_A \underline{u}} \leq \tilde{\mu}_n \quad \forall \underline{u} \in I\!R^n \setminus \{\underline{0}\},$$

and are independent of the discretisation parameter h and the coefficient function $\sigma(\vec{x})$. By a result of Wathen, [14], it is sufficient to consider the *element* matrices. Denoting by λ_{min}^k and λ_{max}^k the minimum and maximum eigenvalues of the diagonally preconditioned element matrix $\left(D_A^k\right)^{-1} A^k$, associated with triangle k, we have,

$$\min_k \left\{\lambda_{min}^k\right\} \leq \tilde{\mu}_1, \quad \tilde{\mu}_n \leq \max_k \left\{\lambda_{max}^k\right\}.$$

For simple geometries, and uniform meshes, we can compute explicit bounds for these values. As an illustration, consider uniform triangles as in Fig. 3.1 and fix normal vectors on each edge as shown.

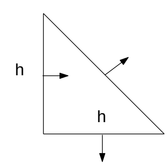

FIG. 3.1. *Uniform triangles with orientated normal vectors*

LEMMA 3.4. *Consider a square domain tiled with uniform triangles of edge length h. Fix normal vectors as shown in Fig. 3.1. Then,*

$$\frac{1}{2} \leq \min_k \left\{\lambda_{min}^k\right\}, \quad \max_k \left\{\lambda_{max}^k\right\} \leq \frac{3}{2}. \tag{3.5}$$

Proof. Labelling the Raviart-Thomas velocity degrees of freedom (x_j, y_j) with $j = 1 : 3$, the element velocity basis functions are,

$$\vec{\varphi}_1 = \begin{pmatrix} -\frac{x_3}{h} + \frac{x}{h} \\ -1 - \frac{y_1}{h} + \frac{y}{h} \end{pmatrix}, \vec{\varphi}_2 = \begin{pmatrix} -\frac{x_3\sqrt{2}}{h} + \frac{x\sqrt{2}}{h} \\ -\frac{y_1\sqrt{2}}{h} + \frac{y\sqrt{2}}{h} \end{pmatrix}, \vec{\varphi}_3 = \begin{pmatrix} 1 + \frac{x_3}{h} - \frac{x}{h} \\ \frac{y_1}{h} - \frac{y}{h} \end{pmatrix}.$$

For element k, let

$$\mathcal{A}_c^k = \begin{pmatrix} \sigma(x_c^k, y_c^k) & 0 \\ 0 & \sigma(x_c^k, y_c^k) \end{pmatrix} = \begin{pmatrix} \sigma & 0 \\ 0 & \sigma \end{pmatrix}$$

denote $\mathcal{A}$ evaluated at the centroid $\left(x_c^k, y_c^k\right)$ of that element. The element velocity mass matrix is then defined by,

$$A^k(i,j) = \int\int_{\triangle} \left(\mathcal{A}_c^k\right)^{-1} \vec{\varphi}_i \cdot \vec{\varphi}_j \, dxdy \quad i,j = 1:3.$$

Integrating over triangle k yields,

$$A^k = h^2 \begin{pmatrix} \frac{1}{3\sigma} & 0 & \frac{1}{6\sigma} \\ 0 & \frac{1}{3\sigma} & 0 \\ \frac{1}{6\sigma} & 0 & \frac{1}{3\sigma} \end{pmatrix}, \quad \left(D_A^k\right)^{-1} A^k = \begin{pmatrix} 1 & 0 & \frac{1}{2} \\ 0 & 1 & 0 \\ \frac{1}{2} & 0 & 1 \end{pmatrix}. \tag{3.6}$$

Hence, $\lambda_1^k = \frac{1}{2}$, $\lambda_2^k = 1$, $\lambda_3^k = \frac{3}{2}$. $\square$

Note that for anisotropic coefficients of the form,

$$\mathcal{A}_c = \begin{pmatrix} \sigma_1(x_c, y_c) & 0 \\ 0 & \sigma_2(x_c, y_c) \end{pmatrix}$$

with $\sigma_1(x_c, y_c) \gg \sigma_2(x_c, y_c)$ or vice versa, diagonal preconditioning for A is not efficient on triangles of this type. The diagonally preconditioned element matrix (3.6) has a zero eigenvalue in the limit of the anisotropy. Rectangular Raviart-Thomas finite elements should be used in that case. On square elements of edge length h, it can be shown that $\lambda_1^k = \frac{1}{2}$, $\lambda_2^k = \frac{1}{2}$, $\lambda_3^k = \frac{3}{2}$, $\lambda_4^k = \frac{3}{2}$, independently of $\sigma_1(\vec{x})$ and $\sigma_2(\vec{x})$.

The following numerical examples illustrate the optimality and robustness of the preconditioner (3.2). We apply one V-cycle ($P_S = S_V$) of `amg1r5` (implemented with symmetric Gauss-Seidel smoothing) to the sparse matrix $S_D = BD_A^{-1}B^T$. We implement it as a black-box; no parameters are estimated a-priori.

4. Numerical Examples. First, consider (2.1) with $\sigma(\vec{x}) = 1$, (unit coefficients), $f = 1$, $g = 0$ and $\partial\Omega_N = \emptyset$, discretised on $\Omega = [0,1] \times [0,1]$. We apply preconditioned MINRES to the assembled system with a stopping tolerance of 10^{-6} on the relative residual error in the $\|\cdot\|_{P^{-1}}$ norm. Iteration counts are reported in Table 4.1. The second column corresponds to unpreconditioned MINRES; the third column lists counts for the exact version of the preconditioner. The time units reported in parentheses are elapsed time in seconds for the total solve using a mex fortran interface in Matlab 6.0 on a SUN ultraSPARC workstation. The symbol * signifies that more than 500 iterations were needed.

TABLE 4.1
MINRES *iterations and (time), TOL* $= 10^{-6}$

h	$P = I$	$P_S = S_D$	$P_S = S_V$
$\frac{1}{16}$	186	26 (0.56)	26 (0.18)
$\frac{1}{32}$	375	26 (2.45)	26 (0.48)
$\frac{1}{64}$	*	26 (14.37)	26 (1.90)
$\frac{1}{128}$	*	26 (84.60)	26 (9.06)

The iteration counts with AMG preconditioning are independent of h and match the iteration counts for the exact version of the preconditioner. We observe that solve times grow linearly with respect to the problem size and a significant time saving is achieved compared to an exact implementation of the preconditioner. The eigenvalues of the preconditioned matrix with $P_S = S_V$ are shown in Table 4.2. We have $\theta^2 \approx 0.954$ and $\Theta^2 = 1$. Substituting these values in Corollary 3.3 gives the theoretical bound $[-0.781, -0.477] \cup [0.5, 2]$.

TABLE 4.2
Eigenvalues of indefinite preconditioned system; $P_S = S_V$

h	$\tilde{\mu}_1$	$\tilde{\mu}_n$	θ^2	Θ^2	Observed eigenvalues
$\frac{1}{8}$	0.5	1.5	0.957	1	$[-0.777,\ -0.532] \cup [0.707,\ 1.896]$
$\frac{1}{16}$	0.5	1.5	0.955	1	$[-0.777,\ -0.508] \cup [0.707,\ 1.939]$
$\frac{1}{32}$	0.5	1.5	0.954	1	$[-0.777,\ -0.496] \cup [0.707,\ 1.964]$

Next we consider a discontinuous coefficient example. Take

$$\sigma(\vec{x}) = \begin{cases} 10^{-3} & \text{in } \Omega_* = [0.25, 0.75] \times [0.25, 1] \\ 1 & \text{in } \Omega \backslash \Omega_* \end{cases}$$

with $f = 0$ and mixed boundary conditions: $\vec{u} \cdot \vec{n} = 0$ on $\partial\Omega_N$, $p = 1 - x$ on $\partial\Omega_D$, where $\partial\Omega_N = [0, 1] \times 0 \cup \{0, 1\} \times [0, 0.75]$ and $\partial\Omega_D = \Omega \backslash \Omega_N$. Iteration counts are given in Table 4.3.

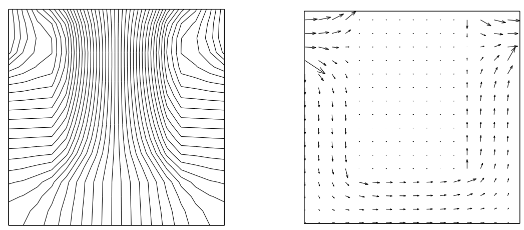

FIG. 4.1. *Pressure contours (left) and velocity field (right)*

TABLE 4.3
MINRES *iterations and (time),* $TOL = 10^{-6}$

h	$P = I$	$P_S = S_D$	$P_S = S_V$
$\frac{1}{16}$	*	25 (0.5)	25 (0.17)
$\frac{1}{32}$	*	25 (2.3)	26 (0.46)
$\frac{1}{64}$	*	25 (12.7)	27 (2.67)
$\frac{1}{128}$	*	25 (81.1)	27 (10.74)

The AMG preconditioner is optimal with repect to the mesh parameter and the solve times grow linearly. Eigenvalues are reported in Table 4.4 and Fig. 4.2 illustrates the spectral equivalence of the exact and inexact versions of the preconditioner. We observe that $\theta^2 \approx 0.835$ and $\Theta^2 = 1$, yielding the theoretical bound $[-0.781, -0.433] \cup [0.5, 2]$ for the preconditioned system.

Note that θ^2 does change with $\sigma(\vec{x})$ but not significantly in this context. It is bounded below by a constant that is independent of h and not close to zero. This property is independent of the jump coefficient (see Table 4.5). Comparing the iteration counts for the two examples, we see that the AMG preconditioner is completely insensitive to the coefficient $\sigma(\vec{x})$.

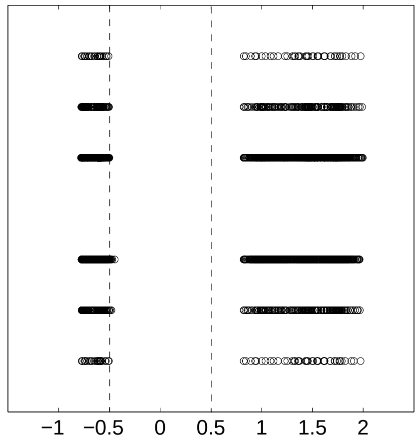

FIG. 4.2. *Eigenvalues of indefinite preconditioned matrix; $P_S = S_D$ (top), $P_S = S_V$ (bottom), $h = \frac{1}{8}, \frac{1}{16}, \frac{1}{32}$*

TABLE 4.4
Eigenvalues of indefinite preconditioned system; $P_S = S_V$

h	$\tilde{\mu}_1$	$\tilde{\mu}_n$	θ^2	Θ^2	Observed eigenvalues
$\frac{1}{8}$	0.5	1.5	0.918	1	$[-0.774, -0.480] \cup [0.822, 1.965]$
$\frac{1}{16}$	0.5	1.5	0.845	1	$[-0.775, -0.448] \cup [0.822, 1.965]$
$\frac{1}{32}$	0.5	1.5	0.836	1	$[-0.775, -0.452] \cup [0.822, 1.974]$

TABLE 4.5
θ^2 and Θ^2 for varying jump coefficient, $h = \frac{1}{16}$,

	10^0	10^{-1}	10^{-2}	10^{-3}	10^{-4}	10^{-5}	10^{-6}
θ^2	0.929	0.868	0.850	0.845	0.845	0.845	0.845
Θ^2	1	1	1	1	1	1	1

As a final example, consider a simple anisotropic test problem. Choose coefficients $\mathcal{A} = \mathtt{diag}\left(10^{-4}, 1\right)$ and homogeneous Dirichlet boundary conditions. Since diagonal scaling for the weighted mass matrix A is now not efficient on triangles, we use uniform rectangular meshes. Iteration counts are given in Table 4.6.

TABLE 4.6
MINRES *iterations and (time), $TOL = 10^{-6}$*

h	$P = I$	$P_S = S_D$		$P_S = S_V$	
$\frac{1}{32}$	*	27	(1.43)	27	(0.31)
$\frac{1}{64}$	*	25	(7.03)	27	(1.29)
$\frac{1}{128}$	*	24	(47.30)	26	(8.68)

We obtain $\tilde{\mu}_1 = 0.5$, $\tilde{\mu}_n = 1.5$, $\theta^2 \approx 0.9477$ and $\Theta^2 = 1$ yielding the theoretical eigenvalue bound $[-0.781, -0.479] \cup [0.5, 2]$ which is observed to be tight.

Note that there is a slight time overhead in the last example. To recover linear growth, we could experiment with the coarsest-grid solver and the size of the coarsest grid problem etc. but this is not in the spirit of a 'black-box' approach. The generic method works equally well for all the coefficients considered. Numerical experiments not reported here also show that variable diagonal coefficients can be treated with equal success.

5. Concluding Remarks. In this paper we have shown that the symmetric indefinite linear systems arising from Raviart-Thomas mixed finite element formulation of diffusion problems can be solved efficiently using a simple generic black-box preconditioning strategy. The preconditioner discussed essentially consists of diagonal scaling for a weighted mass matrix and one V-cycle of AMG applied to a discrete scalar diffusion operator. The resulting method is robust with respect to the PDE coefficients, yields optimal rates of convergence for MINRES and there are no parameters to estimate.

For a discussion of this preconditioning scheme in the context of Stokes problems see [6].

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