NONCONFORMING V-CYCLE AND F-CYCLE MULTIGRID METHODS FOR THE BIHARMONIC PROBLEM USING THE MORLEY ELEMENT

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Abstract. Multigrid V-cycle and F-cycle algorithms for the biharmonic problem using the Morley element are studied in the paper. New results concerning the asymptotic behavior of the contraction numbers of these algorithms with respect to the number of smoothing steps are presented.

1. Introduction

Let $\Omega \subset \mathbb{R}^2$ be a bounded polygonal domain. Consider the following variational problem for the biharmonic equation with homogeneous Dirichlet boundary conditions: Find $u \in H_0^2(\Omega)$ such that

$$a(u,v) = \phi(v) \quad \forall v \in H_0^2(\Omega), \tag{1.1}$$

where

$$a(u,v) = \int_\Omega D^2u : D^2v\,dx := \int_\Omega \sum_{i,j=1}^{2} \frac{\partial^2 u}{\partial x_i \partial x_j} \cdot \frac{\partial^2 v}{\partial x_i \partial x_j}\,dx,$$

and $\phi \in H^{-2}(\Omega) = [H_0^2(\Omega)]'$.

The elliptic regularity of the biharmonic equation (cf. [15] and [17]) implies that there exists $\alpha \in (\frac{1}{2}, 1]$ such that the solution u of (1.1) belongs to $H^{2+\alpha}(\Omega) \cap H_0^2(\Omega)$ whenever $\phi \in H^{-2+\alpha}(\Omega)$ and

$$\|u\|_{H^{2+\alpha}(\Omega)} \le C_\Omega \|\phi\|_{H^{-2+\alpha}(\Omega)}. \tag{1.2}$$

The problem (1.1) can be solved numerically using the Bogner-Fox-Schmit element (cf. [4]), the Argyris element (cf. [1]), the Hsieh-Clough-Tocher element (cf. [14]), the Morley element (cf. [21]) and the incomplete biquadratic element (cf. [24]). Here we will concentrate on the Morley element, which is the simplest among all the finite element methods for the biharmonic problem.

Multigrid methods for the Morley element, which is nonconforming, have been studied in [7], [8], [9], [19], [23], [22] and [25]. It was shown in [9] that the W-cycle multigrid method converges uniformly if the number of smoothing steps is large enough and that the symmetric variable V-cycle algorithm is an optimal preconditioner, without assuming full elliptic regularity.

In [11] and [10], an additive theory was developed to study the asymptotic behavior of the contraction numbers of V-cycle and F-cycle methods for conforming

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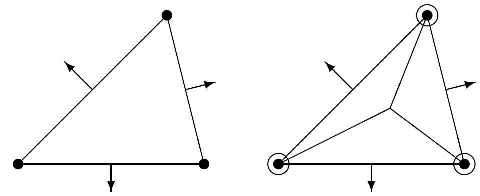


FIGURE 1. The Morley element and the H-C-T element

and nonconforming finite elements for second order problems, without assuming full elliptic regularity. In this paper we will apply this theory to the biharmonic problem. We use the Morley element to illustrate the theory, which can also be applied to other elements (cf. [28]).

Let $\gamma_{k,m}$ be the contraction number of the k-th level symmetric V-cycle algorithm with m pre-smoothing and m post-smoothing steps. Our main result states that, there exists a constant C, independent of k and m, such that

$$\gamma_{k,m} \leq \frac{C}{m^{\alpha/2}} \quad \text{for } m \geq m_0,$$

where the positive integer m_0 is also independent of k. A similar result also holds for the F-cycle algorithm.

The rest of the paper is organized as follows. We discuss the Morley element and its relation with the Hsieh-Clough-Tocher element in Section 2. Some known results concerning the Morley element are summarized in Section 3. Multigrid V-cycle and F-cycle algorithms and the statements of the main results are given in Section 4. Numerical results are presented in Section 5.

2. The Morley element and the Hsieh-Clough-Tocher element

Let $(K, \mathcal{P}, \mathcal{N})$ be the Morley finite element. Then K is a triangle, the set of shape functions $\mathcal{P}$ consists of all quadratic polynomials on K and the set $\mathcal{N}$ of nodal variables consists of the evaluations of the shape functions at the vertices of K and the evaluations of the normal derivatives at the midpoints of the edges of K. See Figure 1(a).

Let $(K, \tilde{\mathcal{P}}, \tilde{\mathcal{N}})$ be the Hsieh-Clough-Tocher macro element (See Figure 1(b)). Then $\tilde{\mathcal{P}}$ is the set of all C^1 functions on K whose restriction to each smaller triangle formed by connecting the centroid and two vertices of K is a cubic polynomial. The nodal variables include the evaluations of the shape functions at the vertices of K, the evaluations of the gradients at the vertices and of the normal derivatives at the midpoints of the edges of K.

Since $\mathcal{P} \subset \tilde{\mathcal{P}}$ and $\mathcal{N} \subset \tilde{\mathcal{N}}$, we call the Hsieh-Clough-Tocher element a "relative" of the Morley element (cf. [9]).

Let $\{\mathcal{T}_k\}_{k\geq 1}$ be a family of triangulations of Ω, where $\mathcal{T}_{k+1}$ is obtained by connecting the midpoints of the edges of the triangles in $\mathcal{T}_k$. We denote the mesh size of $\mathcal{T}_k$ by $h_k = \max\{\text{diam} T : T \in \mathcal{T}_k\}$.

Let V_k be the Morley finite element space associated with $\mathcal{T}_k$. Then $v \in V_k$ if and only if it has the following three properties:

(i) $v_T = v|_T$ is a quadratic polynomial for all $T \in \mathcal{T}_k$,

(ii) v is continuous at the vertices of $\mathcal{T}_k$ and vanishes at the vertices along $\partial\Omega$,

(iii) The normal derivative $\partial v/\partial n$ is continuous at the midpoints of interelement boundaries and vanishes at the midpoints along $\partial\Omega$.

Note that $V_k \not\subset H_0^2(\Omega)$ (i.e., nonconforming) and $V_{k-1} \not\subset V_k$ (i.e., nonnested).

Let $\tilde{V}_k$ be the Hsieh-Clough-Tocher macro element space associated with $\mathcal{T}_k$. Then a function $\tilde{v} \in \tilde{V}_k$ is C^1 on $\overline{\Omega}$, its restriction to each $T \in \mathcal{T}_k$ is a piecewise cubic function, and its nodal values along $\partial\Omega$ are zeros. Note that $\tilde{V}_k \subset H_0^2(\Omega)$ (i.e. conforming).

Now we discuss the relation between the Morley space and the Hsieh-Clough-Tocher space. We can define an operator $E_k : V_k \longrightarrow \tilde{V}_k$. For each $v \in V_k$, the function $E_k v \in \tilde{V}_k$ is defined as follows. For any internal vertex p and internal midpoint m,

$$\begin{aligned}(E_k v)(p) &= v(p)\\ \frac{\partial(E_k v)}{\partial n}(m) &= \frac{\partial v}{\partial n}(m)\\ (\partial^\beta(E_k v))(p) &= \text{ average of } (\partial^\beta v_i)(p)\end{aligned}$$

where $\beta = (0,1)$ or $(1,0)$, and $v_i = v|_{T_i}$ for T_i with p as a vertex.

We can also define an operator $F_k : \tilde{V}_k \longrightarrow V_k$ as follows. For each $\tilde{v} \in \tilde{V}_k$, $F_k\tilde{v}$ is the function in V_k satisfying

$$(F_k\tilde{v})(p) = \tilde{v}(p) \quad \text{and} \quad \frac{\partial(F_k\tilde{v})}{\partial n}(m) = \frac{\partial\tilde{v}}{\partial n}(m)$$

for every internal vertex p and midpoint m of $\mathcal{T}_k$.

The operators E_k and F_k satisfy the following two properties (cf. [9]).

$$F_k \circ E_k = Id_k, \tag{2.1}$$

$$\|F_k\tilde{v}\|_{L_2(\Omega)} \lesssim \|\tilde{v}\|_{L_2(\Omega)} \quad \text{and} \quad \|F_k\tilde{v}\|_{a_k} \lesssim |\tilde{v}|_{H^2(\Omega)} \quad \forall \tilde{v} \in \tilde{V}_k, \tag{2.2}$$

where Id_k is the identity operator on V_k,

$$\|v\|_{a_k} = a_k(v,v)^{1/2} \qquad \forall v \in H_0^2(\Omega) + V_k,$$

and the bilinear form $a_k(\cdot,\cdot)$ on $H_0^2(\Omega) + V_k$ is defined by

$$a_k(u,v) := \sum_{T\in\mathcal{T}^k} \int_T D^2u : D^2v\,dx.$$

Note that the constructions of E_k and F_k and the properties (2.1) and (2.2) rely on the fact that the Morley element and the Hsieh-Clough-Tocher element are relatives.

Now we define the Morley element method and the modified Morley element method for (1.1).

If $\phi(v) = \int_\Omega fv\,dx$ for a function $f \in L_2(\Omega)$, then the Morley finite element method for (1.1) is: Find $u_k \in V_k$ such that

$$a_k(u_k, v) = \int_\Omega fv\,dx \quad \forall v \in V_k. \tag{2.3}$$

In the case where $\phi \in H^{-2}(\Omega)$, the modified Morley finite element method for (1.1) is: Find $u_k' \in V_k$ such that

$$a_k(u_k', v) = \phi(E_k v) \quad \forall v \in V_k. \tag{2.4}$$

The properties of these methods are discussed in [9].

3. Some known results for the Morley space

Let the discrete inner product $(\cdot,\cdot)_k$ on V_k be defined by

$$(v_1, v_2)_k := h_k^2 \left[\sum_{p\in\mathcal{V}_k} n(p)v_1(p)v_2(p) + \sum_{e\in\mathcal{E}_k} \int_e \left(\frac{\partial v_1}{\partial n}\, ds\right)\left(\int_e \frac{\partial v_2}{\partial n}\, ds\right)\right],$$

where $\mathcal{V}_k$ is the set of internal vertices of $\mathcal{T}_k$, $\mathcal{E}_k$ is the set of internal edges of $\mathcal{T}_k$ and $n(p) = \frac{1}{6}\times$ the number of triangles sharing the node p as a vertex. We can represent the bilinear form $a_k(\cdot,\cdot)$ by the operator $A_k : V_k \longrightarrow V_k$ defined by

$$(A_k v_1, v_2)_k = a_k(v_1, v_2) \quad \forall v_1, v_2 \in V_k. \tag{3.1}$$

Then equations (2.3) and (2.4) can both be rewritten as

$$A_k u_k = f_k, \tag{3.2}$$

where $f_k \in V_k$ is defined by $(f_k, v)_k = \int_\Omega f v\, dx$ for all $v \in V_k$ for the standard Morley method, and $(f_k, v)_k = \phi(E_k v)$ for all $v \in V_k$ for the modified Morley method.

One of the main tools for the convergence analysis of multigrid methods is the mesh-dependent norm $|\!|\!| \cdot |\!|\!|_{s,k}$ (cf. [3]): For each $v \in V_k$ we define

$$|\!|\!| v |\!|\!|_{s,k} = \sqrt{(A_k^{s/2} v, v)_k}. \tag{3.3}$$

It is easy to see that

$$|\!|\!| v |\!|\!|_{0,k}^2 = (v, v)_k \approx \|v\|_{L_2(\Omega)} \qquad \forall v \in V_k \tag{3.4}$$

$$|\!|\!| v |\!|\!|_{2,k} = \|v\|_{a_k} \lesssim h_k^{-2} \|v\|_{L_2(\Omega)} \quad \forall v \in V_k. \tag{3.5}$$

To avoid the proliferation of constants we use two notations $\lesssim$ and $\approx$. The statement $A \lesssim B$ means that A is bounded by B multiplied by a constant which is independent of mesh sizes and all arguments in A and B, and $A \approx B$ means $A \lesssim B$ as well as $B \lesssim A$.

In order to describe multigrid methods, we need to define the intergrid transfer operators. We first define $I_{k-1}^k : V_{k-1} \longrightarrow V_k$, the coarse-to-fine intergrid transfer operator.

Let $v \in V_{k-1}$. We define $I_{k-1}^k v \in V_k$ by an averaging technique as follows:

1. If p is an internal vertex of $\mathcal{T}_k$, then

$$I_{k-1}^k v(p) = \frac{1}{|S_{p,k-1}|} \sum_{T\in S_{p,k-1}} v_T(p),$$

where $S_{p,k-1} := \{T \in \mathcal{T}_{k-1} : p \in \partial T\}$.

2. If e is an internal edge of $\mathcal{T}_k$, which means that $e \subset \partial T_1 \cap \partial T_2$ for some $T_1, T_2 \in \mathcal{T}_k$, then

$$\int_e \frac{\partial I_{k-1}^k v}{\partial n}\, ds = \frac{1}{2}\left(\int_e \frac{\partial v_{T_1}}{\partial n}\, ds + \int_e \frac{\partial v_{T_2}}{\partial n}\, ds\right).$$

We can then define the fine-to-coarse operator $I_k^{k-1} : V_k \longrightarrow V_{k-1}$ and the nonconforming Ritz "projection" operator $P_k^{k-1} : V_k \longrightarrow V_{k-1}$ as follows:

$$(I_{k-1}^k v, w)_k = (v, I_k^{k-1} w)_{k-1} \qquad \forall v \in V_{k-1}, w \in V_k,$$
$$a_k(I_{k-1}^k v, w) = a_{k-1}(v, P_k^{k-1} w) \quad \forall v \in V_{k-1}, w \in V_k.$$

Let $\zeta \in H^{2+\alpha}(\Omega) \cap H_0^2(\Omega)$, $\zeta_k \in V_k$ and $\zeta_{k-1} \in V_{k-1}$ be related by

$$a(\zeta, E_k v) = a_k(\zeta_k, v) \qquad \forall v \in V_k,$$
$$a(\zeta, E_{k-1} v) = a_{k-1}(\zeta_{k-1}, v) \quad \forall v \in V_{k-1}.$$

Then the following estimates were established in [9].

$$\|\zeta - \zeta_k\|_{a_k} \lesssim h_k^\alpha \|\zeta\|_{H^{2+\alpha}(\Omega)}, \tag{3.6}$$
$$|||\Pi_k \zeta - \zeta_k|||_{2-\alpha,k} \lesssim h_k^{2\alpha} \|\zeta\|_{H^{2+\alpha}(\Omega)}, \tag{3.7}$$
$$|||\zeta_{k-1} - P_k^{k-1} \zeta_k|||_{2-\alpha,k-1} \lesssim h_k^{2\alpha} \|\zeta\|_{H^{2+\alpha}(\Omega)}, \tag{3.8}$$

where α is the index of elliptic regularity in (1.2) and $\Pi_k : H_0^2(\Omega) \longrightarrow V_k$ is the Morley interpolation operator defined as follows. For each $\zeta \in H_0^2(\Omega)$, the function $\Pi_k \zeta \in V_k$ satisfies

$$(\Pi_k v)(p) = v(p) \quad \text{and} \quad \int_e \frac{\partial(\Pi_k v)}{\partial n}\, ds = \int_e \frac{\partial v}{\partial n}\, ds,$$

where p and e range over the internal vertices and edges of $\mathcal{T}_k$.

The following estimates concerning I_{k-1}^k and Π_k have also been established in [9] :

$$|||I_{k-1}^k v|||_{s,k} \lesssim |||v|||_{s,k-1} \qquad \forall v \in V_{k-1},\ 0 \le s \le 2, \tag{3.9}$$
$$\|\zeta - \Pi_k \zeta\|_{L_2(\Omega)} \lesssim h_k^2 |\zeta|_{H^2(\Omega)} \qquad \forall \zeta \in H_0^2(\Omega), \tag{3.10}$$
$$\|\zeta - \Pi_k \zeta\|_{a_k} \lesssim h_k^\alpha \|\zeta\|_{H^{2+\alpha}} \qquad \forall \zeta \in H^{2+\alpha}(\Omega) \cap H_0^2(\Omega), \tag{3.11}$$
$$|||\Pi_k \zeta - I_{k-1}^k \Pi_{k-1} \zeta|||_{2-\alpha,k} \lesssim h_k^{2\alpha} \|\zeta\|_{H^{2+\alpha}(\Omega)} \quad \forall \zeta \in H^{2+\alpha}(\Omega) \cap H_0^2(\Omega). \tag{3.12}$$

4. V-cycle and F-cycle Multigrid methods

In this section we describe the V-cycle and F-cycle multigrid methods for the Morley finite element and state the main results.

Symmetric V-cycle Multigrid Method (cf. [5], [7], [13], [16], [18], [26] and [27]) The symmetric V-cycle multigrid algorithm is an iterative solver for equations of the form of (3.2). Given $g \in V_k$ and initial guess $z_0 \in V_k$, the output $MG_{\mathcal{V}}(k, g, z_0, m)$ of the algorithm is an approximate solution for the equation

$$A_k z = g, \tag{4.1}$$

where m is the number of pre-smoothing and post-smoothing steps.

For $k = 1$, we define

$$MG_{\mathcal{V}}(1, g, z_0, m) = A_1^{-1} g.$$

For $k \ge 2$, we obtain $MG_{\mathcal{V}}(k, g, z_0, m)$ in three steps.

1. (Pre-Smoothing) For $j = 1, 2, \cdots, m$, compute z_j by

$$z_j = z_{j-1} + \frac{1}{\Lambda_k}(g - A_k z_{j-1}),$$

where Λ_k is a constant dominating the spectral radius of A_k.

2. (Coarse Grid Correction) Compute z_{m+1} by

$$z_{m+1} = z_m + I_{k-1}^k MG_{\mathcal{V}}(k-1, I_k^{k-1}(g - A_k z_m), 0, m).$$

3. (Post-Smoothing) For $j = m+2, \cdots, 2m+1$, compute z_j by

$$z_j = z_{j-1} + \frac{1}{\Lambda_k}(g - A_k z_{j-1}).$$

Finally we set $MG_{\mathcal{V}}(k, g, z_0, m)$ to be z_{2m+1}.

In this algorithm we use Richardson relaxation as the smoother for simplicity. Other smoothers can also be used. (cf. [2], [6] and [12])

F-cycle Multigrid Method (cf. [20], [27], and [26]) The k-th level F-cycle algorithm (associated with the symmetric V-cycle algorithm) produces an approximate solution $MG_{\mathcal{F}}(k, g, z_0, m)$ for (4.1). For $k = 1$, we define

$$MG_{\mathcal{F}}(1, g, z_0, m) = A_1^{-1} g.$$

For $k \geq 2$, we obtain $MG_{\mathcal{F}}(k, g, z_0, m)$ in three steps.

1. (Pre-Smoothing) For $j = 1, 2, \cdots, m$, compute z_j by

$$z_j = z_{j-1} + \frac{1}{\Lambda_k}(g - A_k z_{j-1}).$$

2. (Coarse Grid Correction) Compute $z_{m+\frac{1}{2}}$ and z_{m+1} by

$$\begin{aligned} z_{m+\frac{1}{2}} &= MG_{\mathcal{F}}(k-1, I_k^{k-1}(g - A_k z_m), 0, m). \\ z_{m+1} &= z_m + I_k^{k-1} MG_{\mathcal{V}}(k-1, I_k^{k-1}(g - A_k z_m), z_{m+\frac{1}{2}}, m). \end{aligned}$$

3. (Post-Smoothing) For $j = m+2, \cdots, 2m+1$, compute z_j by

$$z_j = z_{j-1} + \frac{1}{\Lambda_k}(g - A_k z_{j-1}).$$

Finally we set $MG_{\mathcal{F}}(k, g, z_0, m)$ to be z_{2m+1}.

Let $\mathbb{E}_{k,m} : V_k \longrightarrow V_k$ be the error propagation operator of the symmetric V-cycle algorithm applied to the equation (4.1), i.e.,

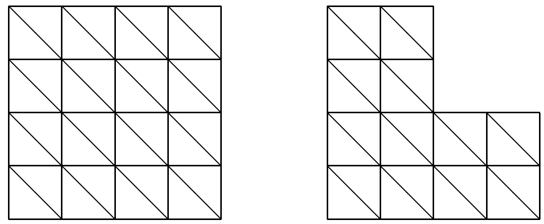
$$\mathbb{E}_{k,m}(z - z_0) = z - MG_{\mathcal{V}}(k, g, z_0, m),$$

where z is the exact solution of (4.1). The starting point of the additive theory is the following additive expression for $\mathbb{E}_{k,m}$ (cf. [10]).

$$\begin{aligned} \mathbb{E}_{k,m} = \sum_{j=2}^{k} R_k^m I_{k-1}^k \cdots R_{j+1}^m I_j^{j+1} R_j^m (Id_j - I_{j-1}^j P_j^{j-1}) R_j^m \\ \times P_{j+1}^j R_{j+1}^m \cdots P_k^{k-1} R_k^m, \end{aligned} \tag{4.2}$$

where $R_j = Id_j - \Lambda^{-1} A_j$. To apply this theory to nonconforming finite elements, we need the estimates (2.1), (2.2), (3.4), (3.5), (3.6)−(3.12), plus the following new estimates (cf. [28]) on the relation of mesh-dependent norms between two consecutive levels. First of all, we have

$$|\!|\!| I_{k-1}^k v |\!|\!|_{0,k}^2 \leq (1+\theta^2) |\!|\!| v |\!|\!|_{0,k-1}^2 + C_0 \theta^{-2} h_k^{2\alpha} |\!|\!| v |\!|\!|_{\alpha,k-1}^2 \quad \forall v \in V_k,\ \theta \in (0,1), \tag{4.3}$$



(a) Square domain (b) L-shaped domain

FIGURE 2. The triangulation $\mathcal{T}_k$ for $k = 2$.

where the positive constant C_0 is mesh-independent. Moreover, the operator Π_{k-1} can be extended to map $H_0^2(\Omega) + V_k$ to V_{k-1} and we have

$$\|\Pi_{k-1} v\|_{a_k} \lesssim \|v\|_{a_k} \qquad \forall v \in H_0^2(\Omega) + V_k, \tag{4.4}$$

$$\|\Pi_{k-1} v - v\|_{L_2(\Omega)} \lesssim h_k^2 \|v\|_{a_k} \quad \forall v \in V_k, \tag{4.5}$$

$$|\!|\!|\Pi_{k-1} v|\!|\!|_{0,k-1}^2 \le (1+\theta^2)|\!|\!|v|\!|\!|_{0,k}^2 + C_0' \theta^{-2} h_k^{2\alpha} |\!|\!|v|\!|\!|_{\alpha,k}^2 \quad \forall v \in V_k,\ \theta \in (0,1), \tag{4.6}$$

where the positive constant C_0' is mesh-independent.

We can prove the following theorems (cf. [28]).

Theorem 4.1. *There exist a positive constant C and a positive integer m_0, both independent of k, such that for all $m \ge m_0$ and $z_0 \in V_k$,*

$$\|z - MG_{\mathcal{V}}(k, g, z_0, m)\|_{a_k} \le C m^{-\alpha/2} \|z - z_0\|_{a_k}, \tag{4.7}$$

where z is the exact solution of (4.1).

Theorem 4.2. *There exist a positive constant C and a positive integer m_0, both independent of k, such that for all $m \ge m_0$ and $z_0 \in V_k$,*

$$\|z - MG_{\mathcal{F}}(k, g, z_0, m)\|_{a_k} \le C m^{-\alpha/2} \|z - z_0\|_{a_k}, \tag{4.8}$$

where z is the exact solution of (4.1).

5. Numerical Experiments

In this section we present some experimental results to illustrate Theorem 4.1 and Theorem 4.2.

First let Ω be the unit square $(0, 1) \times (0, 1)$ (see Figure 2(a)). Since the domain Ω is convex, we have full elliptic regularity, i.e., the index α in (1.2) is 1. Let $\gamma_{k,m}$ be the contraction number of the kth level V-cycle iteration with m pre-smoothing and m post-smoothing steps. According to Theorem 4.1, there is a constant C, independent of k and m, such that

$$m^{1/2} \gamma_{k,m} \le C. \tag{5.1}$$

The numerical results in Table 1 are consistent with (5.1). In fact, they seem to indicate that C could be some number less than 10 and (5.1) is valid as long as $m \ge 50$.

$m^{1/2}\gamma_{m,k}$	m=20	m=30	m=40	m=50	m=60	m=70	m=80
k=3	1.1347	1.0005	0.9075	0.8251	0.7454	0.6683	0.5948
k=4	1.5547	2.2969	2.1645	2.0767	2.0067	1.9455	1.8904
k=5	3.9972	3.5275	3.3043	3.1722	3.0803	3.0009	2.9491
k=6	5.3075	4.5923	4.2589	4.0665	3.9393	3.8459	3.7726
k=7	6.4352	5.4653	5.0207	4.7645	4.5984	4.4800	4.3898
k=8	7.3727	6.1637	5.6143	5.2982	5.0953	4.9528	4.8459

Table 1. V-cycle results on the unit square

Remark 5.1. Note that the condition number of the operator A_k (cf. (3.1) and (3.5)) is of order h_k^{-4} while the condition number for second order problems is of order h_k^{-2}. Therefore the effect of m smoothing steps for fourth order problems is equivalent to the effect of $\sqrt{m}$ smoothing steps for second order problems.

Let $\tilde{\gamma}_{k,m}$ be the contraction number of the kth level F-cycle iteration with m pre-smoothing and m post-smoothing steps. According to Theorem 4.2, there is a constant C, independent of k and m, such that

$$m^{1/2}\tilde{\gamma}_{k,m} \leq C. \tag{5.2}$$

The numerical results in Table 2 are consistent with (5.2) and seem to indicate that $C = 2$ and (5.2) is valid as long as $m \geq 15$.

$m^{1/2}\tilde{\gamma}_{m,k}$	m=10	m=11	m=12	m=13	m=14	m=15	m=16
k=3	1.2132	1.1890	1.1706	1.1524	1.1364	1.1231	1.1071
k=4	1.4359	1.4037	1.3764	1.4097	1.3907	1.3838	1.4097
k=5	1.4310	1.3909	1.4163	1.4140	1.4030	1.4000	1.3933
k=6	1.4057	1.4041	1.3989	1.4017	1.3924	1.3908	1.3905
k=7	1.7918	1.3958	1.4035	1.3841	1.3756	1.3949	1.3759
k=8	5.4706	3.5578	2.4361	1.7541	1.3662	1.3775	1.3700

Table 2. F-cycle results on the unit square

In the case of the L-shaped domain (see Figure 2(b)), the index of elliptic regularity is $\alpha_* = 0.5444837368$. Numerical results for V-cycle and F-cycle algorithms are reported in Table 3 and Table 4, which are also consistent with (5.1) and (5.2).

$m^{\alpha_*/2}\gamma_{m,k}$	m=30	m=40	m=50	m=60	m=70	m=80	m=90
k=3	0.2237	0.1404	0.0879	0.0546	0.0337	0.0208	0.0127
k=4	0.9099	0.7905	0.7137	0.6597	0.6173	0.6833	0.5561
k=5	1.5698	1.3924	1.2888	1.2124	1.1605	1.1192	1.0877
k=6	2.1111	1.8776	1.7316	1.6282	1.5592	1.5073	1.4635
k=7	2.5752	2.2753	2.0991	1.9814	1.8938	1.8276	1.7715
k=8	2.9648	2.6232	2.4243	2.2913	2.1894	2.1138	2.0517

Table 3. V-cycle results on an L-shaped domain

$m^{\alpha_*/2}\tilde{\gamma}_{m,k}$	m=11	m=12	m=13	m=14	m=15	m=16	m=17
k=3	0.5550	0.5266	0.5006	0.4762	0.4532	0.4313	0.4112
k=4	0.8529	0.8459	0.8126	0.8080	0.7965	0.7858	0.7743
k=5	0.8274	0.8092	0.7942	0.7701	0.7646	0.7303	0.7341
k=6	0.8134	0.7958	0.7830	0.7627	0.7515	0.7391	0.7192
k=7	0.8205	0.8038	0.7894	0.7759	0.7601	0.7381	0.7246
k=8	2.0406	1.4087	1.0198	0.7749	0.7449	0.7264	0.7140

TABLE 4. F-cycle results on an L-shaped domain

Remark 5.2. Even though the asymptotic convergence rate for both algorithms is $O(m^{-\alpha/2})$, the performance of the F-cycle algorithm is clearly superior, as demonstrated by the numerical results in Table 5 and Table 6. Similar results also hold for the L-shaped domain. It would be interesting to find a theoretical explanation for this phenomenon.

$\gamma_{m,k}$	m=34	m=35	m=36	m=37	m=38	m=39	m=40	m=41
k=3	0.1648	0.1609	0.1571	0.1535	0.1500	0.1467	0.1435	0.1404
k=4	0.3834	0.3756	0.3683	0.3613	0.3546	0.3483	0.3422	0.3365
k=5	0.5869	0.5746	0.5630	0.5521	0.5417	0.5318	0.5225	0.5135
k=6	0.7605	0.7438	0.7281	0.7133	0.6993	0.6860	0.6734	0.6614
k=7	0.9014	0.8807	0.8613	0.8430	0.8257	0.8093	0.7935	0.7791
k=8	1.0128	0.9887	0.9667	0.9448	0.9247	0.9057	0.8877	0.8707

TABLE 5. Contraction numbers for V-cycle algorithms on the unit square

$\tilde{\gamma}_{m,k}$	m=11	m=12	m=13	m=14	m=15	m=16
k=3	0.3580	0.3379	0.3196	0.3037	0.2900	0.2768
k=4	0.4232	0.3973	0.3910	0.3717	0.3573	0.3524
k=5	0.4194	0.4089	0.3922	0.3750	0.3615	0.3483
k=6	0.4234	0.4038	0.3888	0.3721	0.3591	0.3476
k=7	0.4208	0.4051	0.3839	0.3677	0.3602	0.3440
k=8	1.0727	0.7032	0.4865	0.3651	0.3557	0.3425

TABLE 6. Contraction numbers for F-cycle algorithms on the unit square

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