FAS interpolation and smoother components: A comparative study

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Outline

- Description of FAS and AMGe based on Element Agglomeration.

- Scalability of the FAS-AMGe algorithm and robustness study.

- Comparative study:

  - Operator vs non-operator dependent interpolation study.

  - Interpolation update study.

  - Point vs System smoother study.

Characteristics of the Problem

Given a nonlinear elliptic scalar equation of the form

$$A(u) = f,$$

where

$$A(u) = G(u)u,$$

want to solve numerically its discretized version

$$A^h(u^h) = f^h.$$

Steady state of Reaction Diffusion Equations

- $u(x,t)$ density function in a diffusion medium Ω.

- Conservation equation in steady state.

- Nonlinear constitutive relation :

  $$Flux = -a(u)\nabla u,$$

  and a steady state rate of generation per unit volume of the form

  $$q = g(u)u - f(x),$$

  imply

  $$L(u) = -\nabla \cdot a(u)\nabla u + g(u)u = f(x).$$

Nonlinear Diffusion

- Filtration.
- Phase transition.
- Biochemistry and dynamics of biological groups.
- Radiation Transport.

Galerkin Formulation

For u, v admissible test functions

$$(L(u), v) = (f, v),$$

where

$$\begin{aligned}(L(u), v) &= \int_\Omega a(u)\nabla u \cdot \nabla v \, dx \\ &+ \int_\Omega g(u)uv \, dx,\end{aligned}$$

$$(f, v) = \int_\Omega fv \, dx.$$

Finite Element Formulation

For u^h, v^h admissible test functions

$$(L^h(u^h), v^h)_h = (f^h, v^h)_h,$$

where

$$\begin{aligned}
(f^h, v^h)_h &= \sum_T \int_T f^h v^h \, dx, \\
(L^h(u^h), v^h)_h &= \sum_T \int_T a(u^h) \nabla u^h \cdot \nabla v^h \, dx \\
&+ \sum_T \int_T g(u^h) u^h v^h \, dx, \\
&\approx \sum_T a_T(u^h) \int_T \nabla u^h \cdot \nabla v^h \, dx \\
&+ \sum_T g_T(u^h) \int_T u^h v^h \, dx.
\end{aligned}$$

If T has vertices i, j, k then
$a_T(u) = a(\bar{u}_T)$ and $g_T(u) = g(\bar{u}_T)$, with

$$\bar{u}_T = \frac{1}{3}(u(i) + u(j) + u(k)).$$

Vector Formulation

If $\Psi = (v_1, \cdots, v_N)$ where $v_j = \varphi_j$ then the previous approximation can be written as

$$\sum_T \left(a_T(u^h) \Psi_T^t A_T + g_T(u^h) \Psi_T^t G_T \right) u_T^h,$$

where A_T stiffness and G_T mass matrices.

Discretizations

We consider discretizations with:

- Unstructured triangular finite element mesh.
- 2D geometries.
- Averages to take advantage of the stiffness and mass matrices.

FAS 2-cycle

Want to solve $A^h(u^h) = f^h$.

- Relax $A^h(u^h) = f^h, \nu_1$ times on Ω_h (initial guess) to obtain v^h.
- Given v^h, compute $r^h = f^h - A^h(v^h)$.
- Restrict $r^{2h} = I_h^{2h} r^h$, $v^{2h} = \tilde{I}_h^{2h} v^h$.
- 'Solve' $A^{2h}(u^{2h}) = f^{2h} = r^{2h} + A^{2h}(v^{2h})$.
- Compute $e^{2h} = u^{2h} - v^{2h}$.
- Correct $v^h = v^h + I_{2h}^h e^{2h}$.
- Relax $A^h(u^h) = f^h, \nu_2$ times on Ω_h.

AMGe: Coarse grid Formation

- Grid : Agglomerate elements.

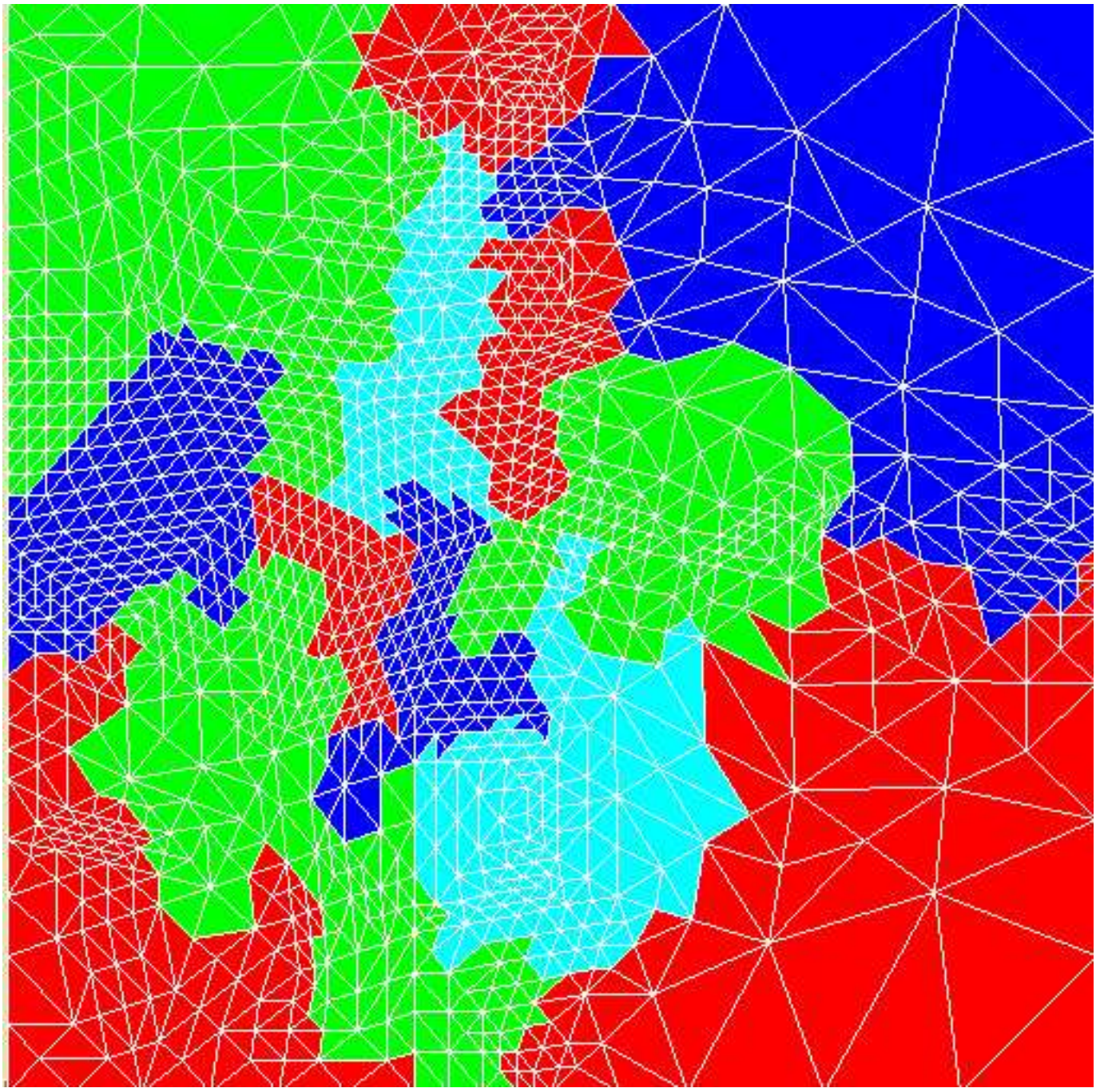
  1. Given element-node and boundary-node relations, create all the relations needed to construct faces as intersections.

  2. Agglomerate elements (coarse elements) to create macro elements by weighting faces (Jones-Vassilevski).

  3. Find coarse grid element-node and boundary-node relations.

  4. Stop if there is only one macro element or if the algorithm have reached the desired number of levels. Otherwise, go back to 1).

Macroelements



AMGe-FAS Method

- V-cycle.

- Grid transfer Operators :

  1. Determine nodes of macro elements (coarse nodes).

  2. Create coarse node-fine node relations (i.e., neighborhoods for interpolation).

  3. Interpolation obtained by minimizing quadratic functional involving neighborhood matrix (Jones-Vassilevski).

- Coarse equations : Local version of fine nonlinearity.

- Relaxation methods : Picard solver, Newton method and Hybrid.

Parameter values in Scalability and Robustness studies

- Domain is the unit square.
- Homogeneous Dirichlet B.C.
- Operator dependent interpolations.
- Interpolations operators are refreshed.
- Only system smoothers are used.
- Nonlinear tolerances are set to 10^{-6}.
- Solution is a multiple of $x(1-x)y(1-y)$.
- Initial guess is constant (some exceptions).

Scalability of FAS-AMGe
FAS-Picard $V(1,1)$

Linear tolerances are 10^{-2} and the maximum number of linear iterations is 1000.

$$a(u) = \frac{1}{\sqrt{u^2 + 10^{-3}}},$$

$$g(u) = 1.$$

Degrees of freedom (levels)	Number of V-cycles (Number of linear iterations)		
	Initial constant guess		
	10	100	1000
1,089 (7)	14 (40)	14 (39)	14 (38)
4,225 (8)	8 (22)	9 (24)	9 (24)
16,641 (9)	5 (13)	6 (13)	8 (17)
66,049 (11)	7 (20)	9 (21)	9 (21)

Scalability of FAS-AMGe
Picard iteration

Linear tolerances are 10^{-2} and the maximum number of linear iterations is 1000.

$$a(u) = \frac{1}{\sqrt{u^2 + 10^{-3}}},$$

$$g(u) = 1.$$

Degrees of freedom (levels)	Number of nonlinear iterations (Number of linear iterations)		
	Initial constant guess		
	10	100	1000
1,089 (7)	13 (106)	29 (110)	31 (112)
4,225 (8)	18 (126)	19 (86)	21 (131)
16,641 (9)	16 (154)	17 (201)	18 (199)
66,049 (11)	15 (724)	16 (718)	16 (80)

Scalability of FAS-AMGe
FAS-Newton $V(1,1)$

Linear tolerances have been set to 10^{-1} and the maximum number of linear iterations is 1000.

$$a(u) = \sqrt{u^2 + u + 1},$$

$$g(u) = 0.$$

Degrees of freedom (levels)	Number of V-cycles (Number of linear iterations)		
	Initial multiple guess		
	13	26	39
1,089 (7)	2 (1)	4 (6)	4 (4)
4,225 (8)	2 (1)	2 (2)	3 (1)
16,641 (9)	2 (2)	2 (2)	3 (4)
66,049 (11)	2 (3)	2 (2)	3 (4)

Scalability of FAS-AMGe
Newton iteration

Linear tolerances have been set to 10^{-1} and the maximum number of linear iterations is 1000.

$$a(u) = \sqrt{u^2 + u + 1},$$

$$g(u) = 0.$$

Degrees of freedom (levels)	Number of nonlinear iterations (Number of linear iterations)		
	Initial multiple guess		
	13	26	39
1,089 (7)	8 (7)	10 (4)	11 (8)
4,225 (8)	8 (13)	10 (13)	11 (15)
16,641 (9)	8 (22)	9 (14)	11 (25)
66,049 (11)	7 (31)	9 (30)	10 (34)

Robustness study

Mesh has 66,049 dofs and 11 levels. Linear tolerances are 10^{-6} and the max. number of linear iterations per nonlinear iteration is 3.

$$a(u) = u^2 + 10^{-3}, \quad g(u) = u.$$

Initial guess	Nonlinear iterations (linear iterations)		Number of V-cycles (Number of finest grid linear iterations)		
	Picard	Newton	FAS-Picard	FAS-Newton	FAS-Hybrid
1.0	15 (45)	10 (30)	1 (6)	5 (30)	2 (12)
5.0	15 (45)	14 (42)	1 (6)	7 (42)	2 (12)
10.0	15 (45)	16 (48)	1 (6)	4 (24)	3 (18)
20.0	13 (39)	18 (54)	1 (6)	17 (102)	3 (18)
50.0	9 (27)	20 (60)	1 (6)	10 (60)	2 (12)
100.0	10 (30)	22 (66)	2 (12)	12 (72)	2 (12)
1000.0	16 (48)	27 (81)	2 (12)	15 (90)	2 (12)
10000.0	8 (24)	33 (99)	4 (24)	diverges	3 (18)
100000.0	17 (51)	39 (117)	diverges	22 (132)	2 (12)

Robustness study

Mesh has 66,049 dofs and 11 levels. Linear tolerances are 10^{-6} and the max. number of linear iterations per nonlinear iteration is 3.

$$a(u) = (u^2 + 10^{-3})^{-1/2}, \quad g(u) = u^2 + 10^{-3}.$$

Initial guess	Nonlinear iterations (linear iterations)		Number of V-cycles (Number of finest grid linear iterations)		
	Picard	Newton	FAS-Picard	FAS-Newton	FAS-Hybrid
0.01	6 (18)	4 (12)	2 (12)	2 (12)	2 (12)
0.03	7 (21)	4 (12)	2 (12)	2 (12)	2 (12)
0.05	8 (24)	4 (12)	2 (12)	2 (12)	2 (12)
0.06	8 (24)	5 (15)	2 (12)	2 (12)	2 (12)
0.07	8 (24)	7 (21)	2 (12)	$>$ 25 (150)	2 (12)
0.08	8 (24)	$>$ 50 (150)	2 (12)	$>$ 25 (150)	2 (12)
1.0	10 (30)	$>$ 50 (150)	2 (12)	$>$ 25 (150)	4 (24)
15.0	9 (27)	$>$ 50 (150)	2 (12)	$>$ 25 (150)	4 (24)
50.0	7 (21)	$>$ 50 (150)	2 (12)	$>$ 25 (150)	2 (12)
100.0	8 (24)	$>$ 50 (150)	2 (12)	$>$ 25 (150)	2 (12)

Conclusions

- Numerical simulations shown scalability of the computational work of the FAS-AMGe algorithm.

- Performance studies indicate nonlinear multigrid schemes appear to be more efficient and robust for poor initial guesses than repeated applications of the nonlinear system smoother.