

Multigrid Acceleration of the Horn-Schunck Algorithm for the Optical Flow Problem

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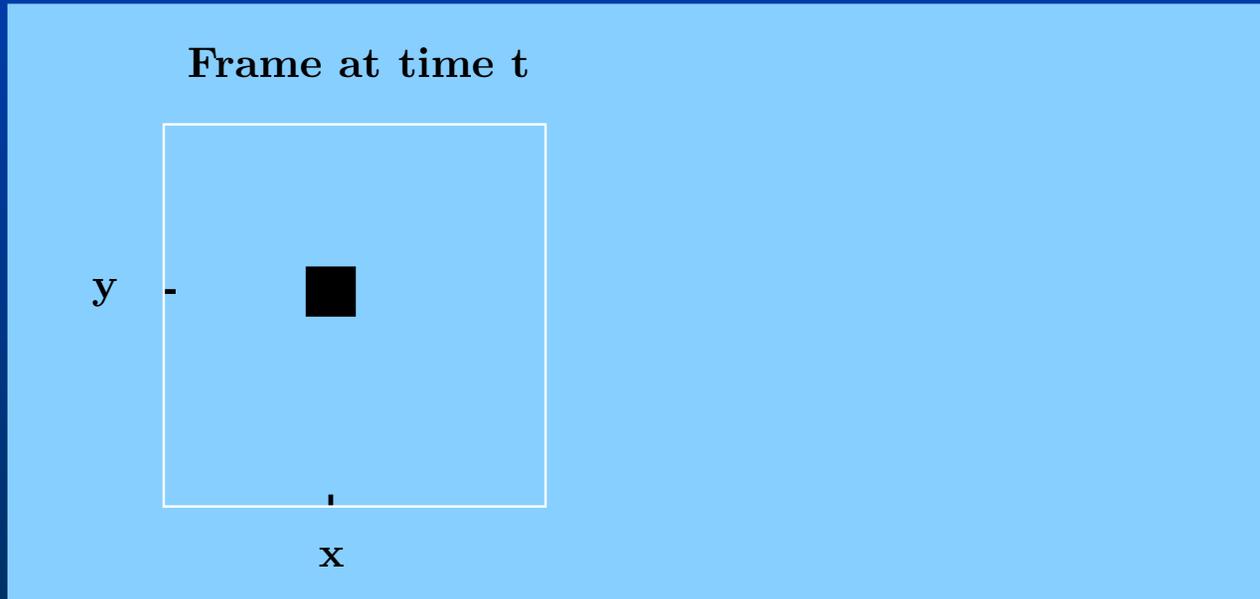
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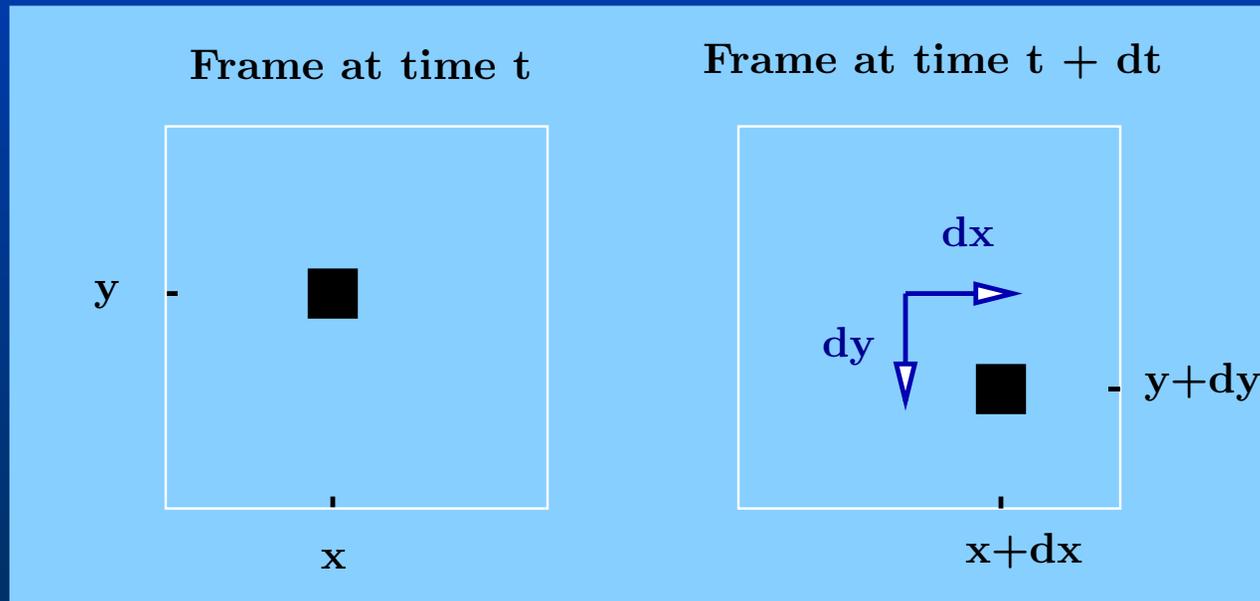
Overview

- Introduction and Related Works
- Optical Flow Constraints
- Regularization
- Horn Schunck Algorithm
- Multigrid Scheme
- A Simple Illustration
- Variational Multigrid
- Experimental Results

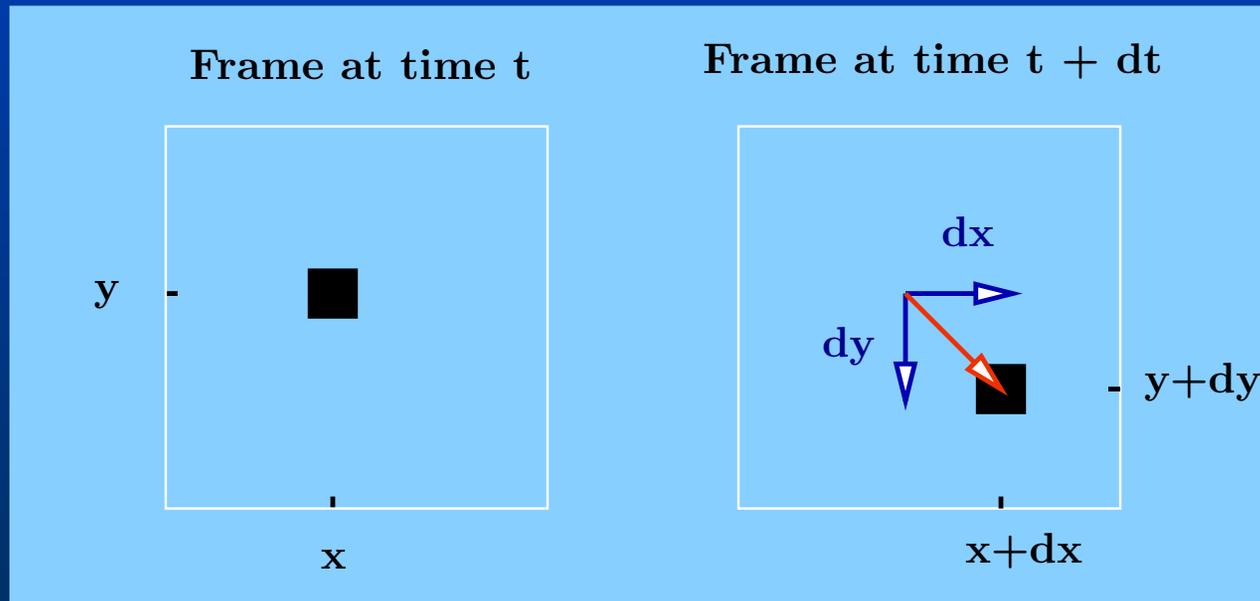
Introduction and Related Works



Introduction and Related Works

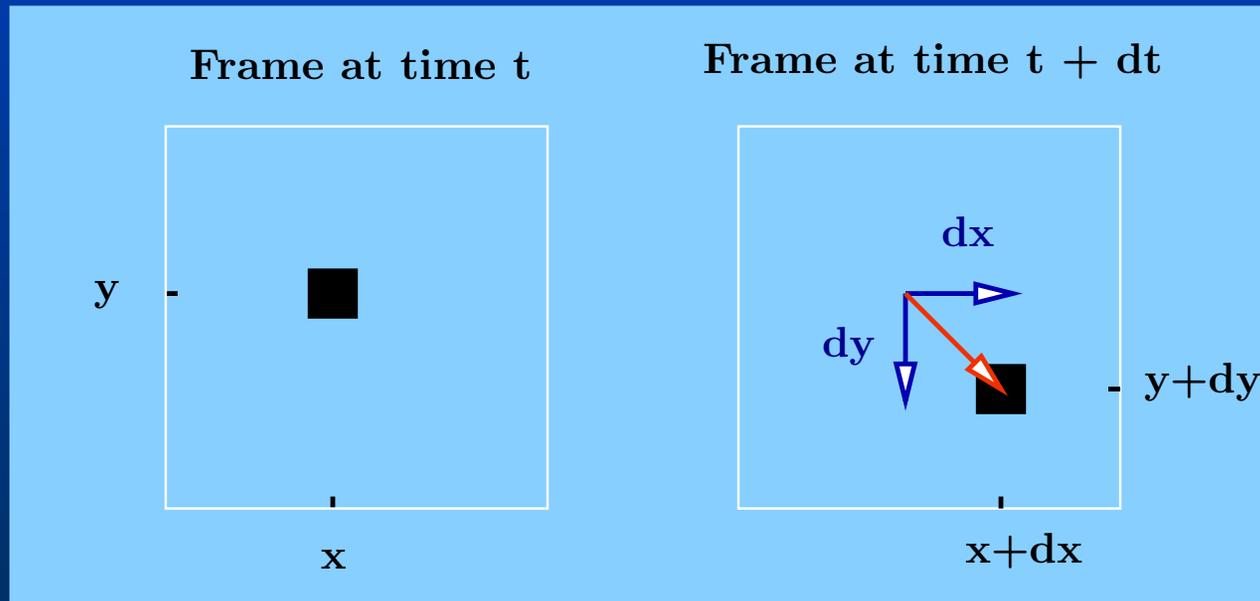


Introduction and Related Works



The optical flow at the pixel (x,y) is the 2D-velocity vector $(u, v) = \left(\frac{dx}{dt}, \frac{dy}{dt}\right)$

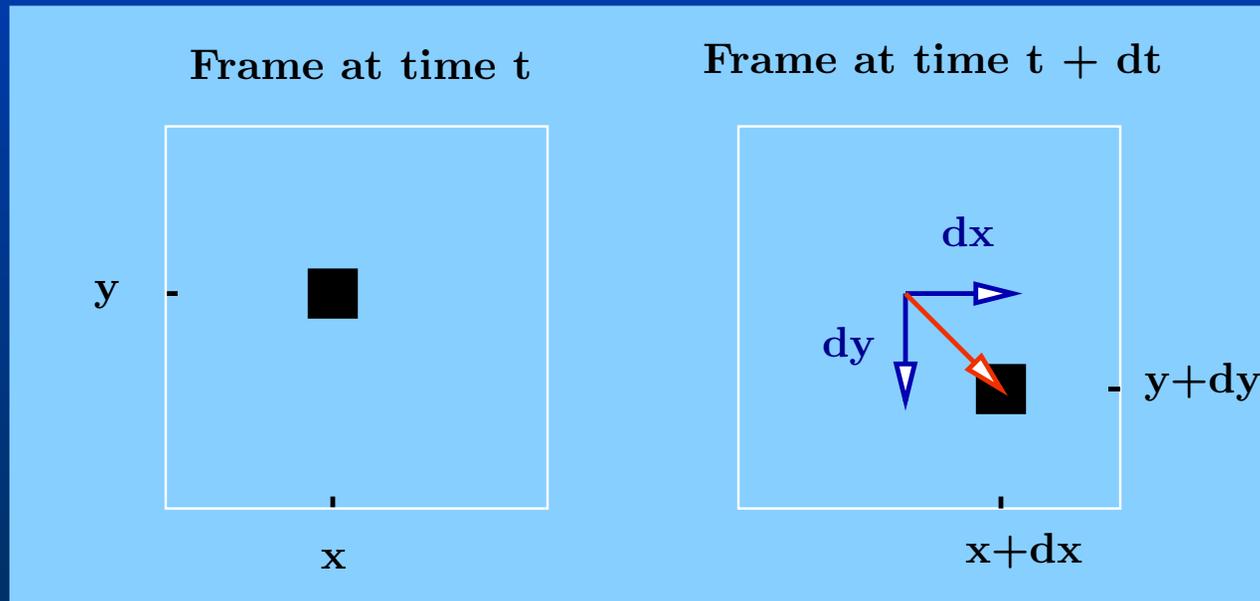
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- An approximation of the 2D-motion

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- An approximation of the 2D-motion
- Optical Flow \neq Motion

Introduction and Related Works

- ✘ Term originates with James Gibson in 1979
- ✘ Quadratic smoothness \leftrightarrow Horn–Schunck (1981)
- ✘ Registration technique with local constraints \leftrightarrow Lucas–Kanade (1981)
- ✘ Oriented smoothness \leftrightarrow Nagel–Enkelmann (1983–86)
- ✘ Multigrid relaxation \leftrightarrow Terzopoulos (1986)
- ✘ Performance evaluation of popular algorithms \leftrightarrow Barron–Fleet–Beauchemin (1994)
- ✘ General anisotropic smoothness \leftrightarrow Weickert (1996)
- ✘ Optimal control framework \leftrightarrow Borzi–Ito–Kunisch (2002)

Optical Flow Constraints

- $I(x, y, t)$: The image intensity of the pixel (x, y) at time t .
- I_x, I_y, I_t : Spatial and temporal derivatives of I .

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Taylor Expansion \implies

$$I_x u + I_y v + I_t = 0$$

The optical flow constraint equation (OFCE)

Optical Flow Constraints

An equivalent form of the (OFCE) :

$$\vec{\nabla}I \cdot \vec{w} = -I_t \Rightarrow \vec{D}I \cdot (\vec{w}, 1) = 0$$

where

$$\vec{\nabla}I = (I_x, I_y) , \quad \vec{D}I = (I_x, I_y, I_t)$$

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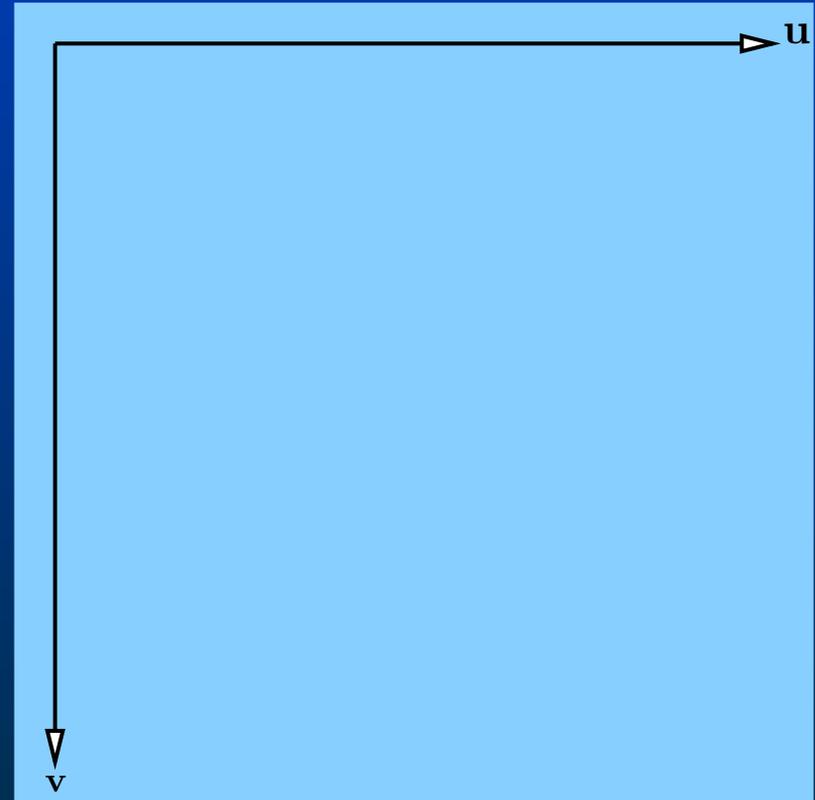
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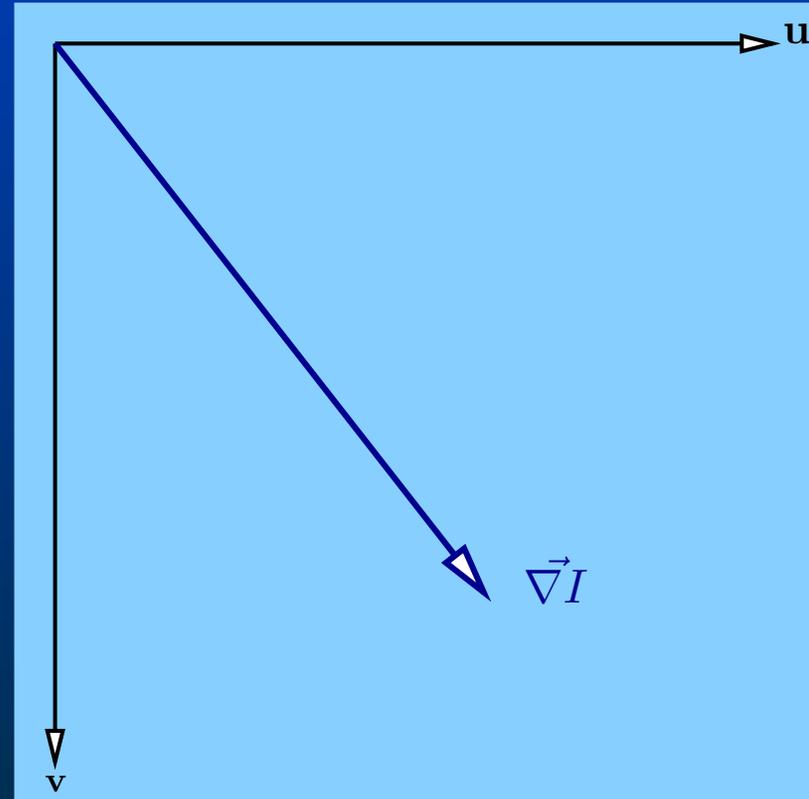
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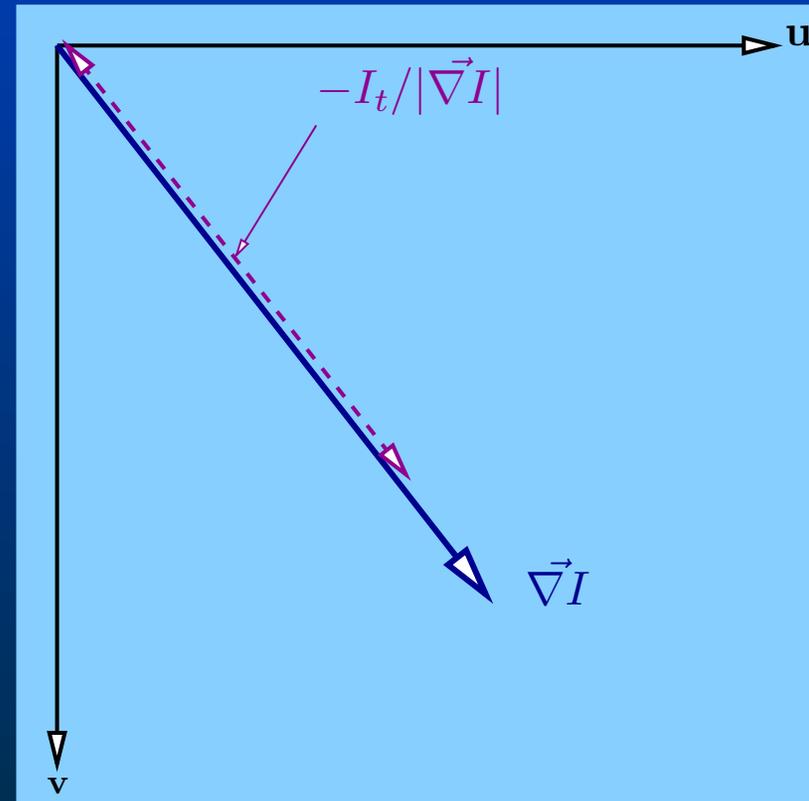
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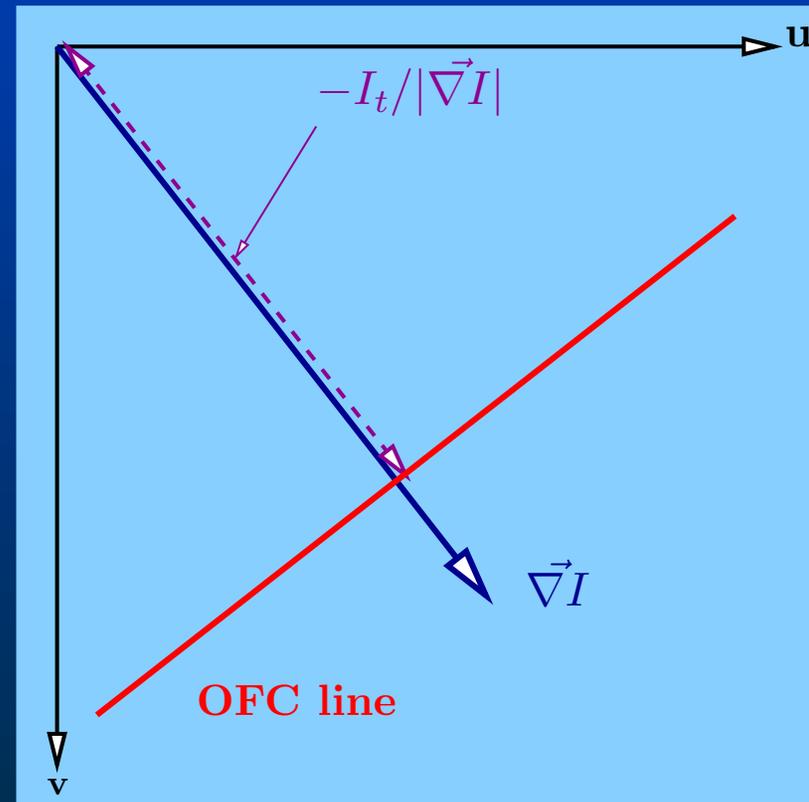
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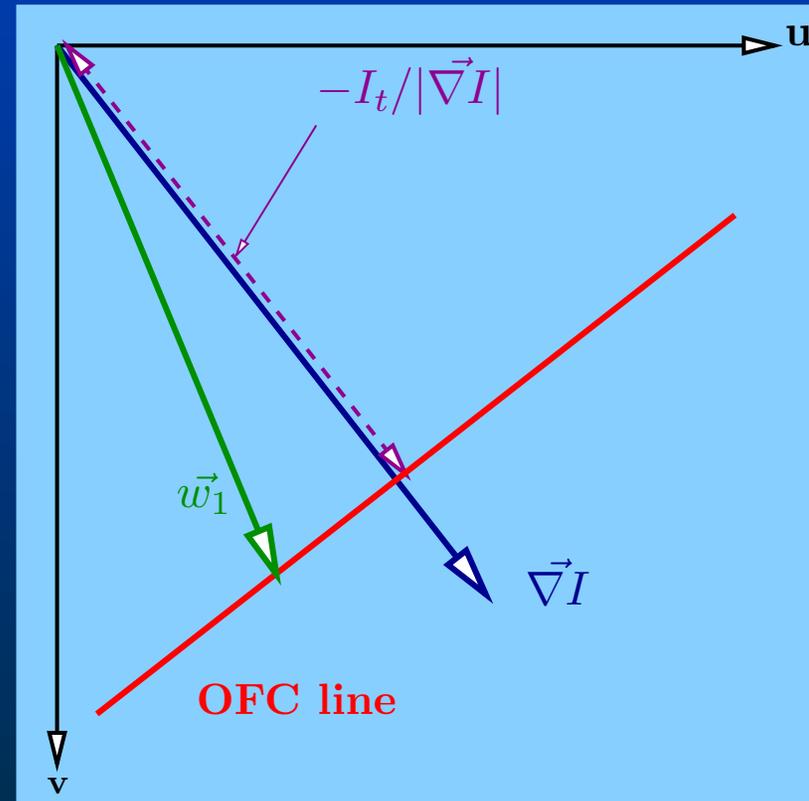
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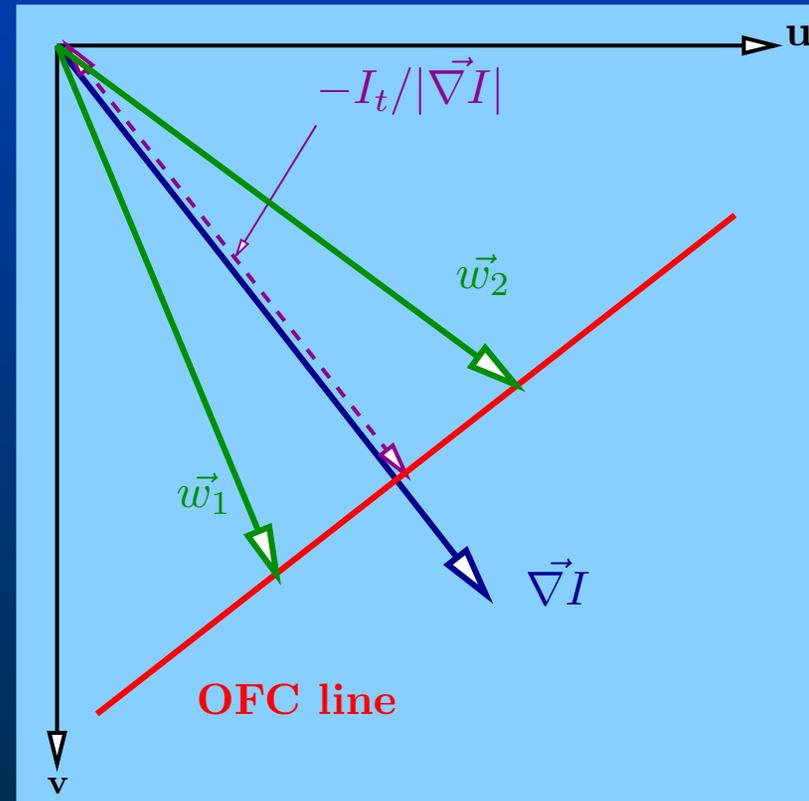
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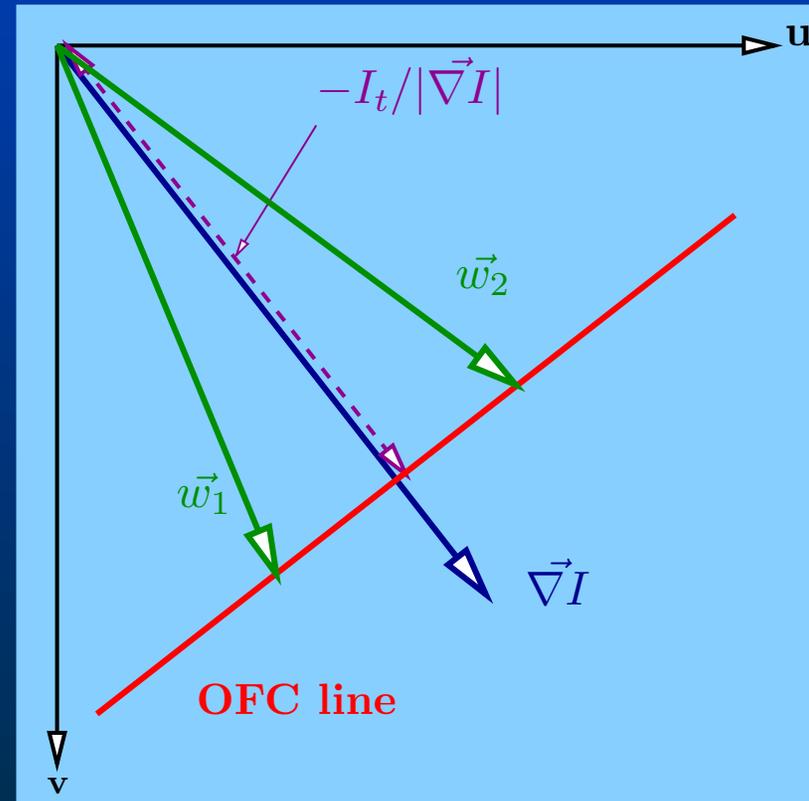
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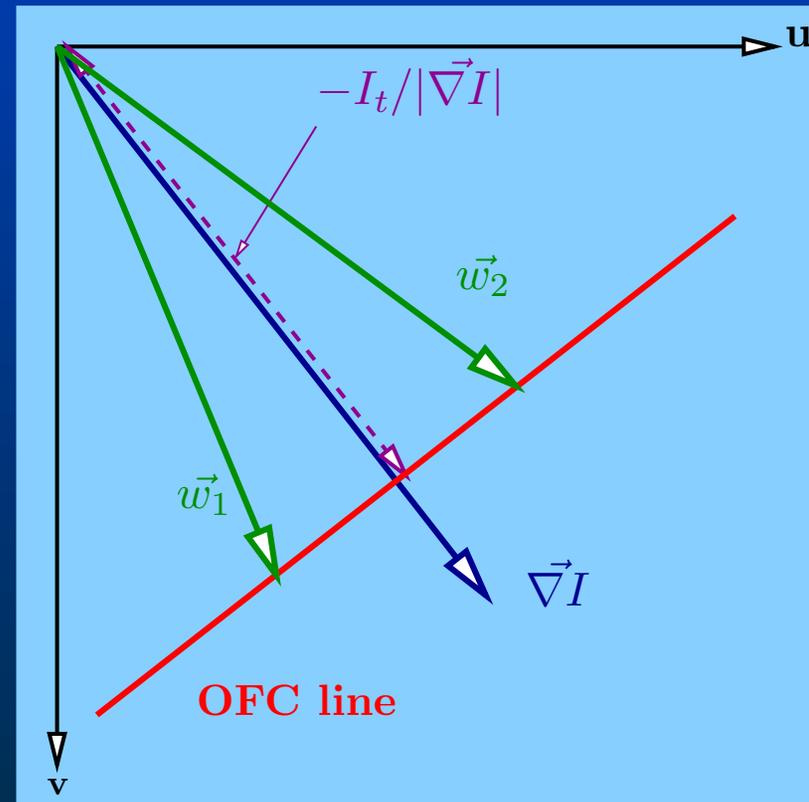
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We can only calculate the normal component of the velocity \vec{w} and not the tangent flow \longrightarrow **The aperture problem.**

Regularization

The (OFCE) is replaced by

$$\min_{(u,v)} \int_x \int_y E(u, v) dx dy \quad (\mathbf{P})$$

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The method for solving (P) will depend on the choice of E_r .

Horn-Schunck Algorithm

A standard choice of E_r is the isotropic stabilizer :

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We discretize the Laplacien Δ by the standard 5 point stencil

$$\begin{pmatrix} & 1 & \\ 1 & -4 & 1 \\ & 1 & \end{pmatrix}$$

and use $\Delta w = \bar{w} - w$ where \bar{w} is the average of the neighbors.

Horn-Schunck Algorithm

Spatial and Temporal Image Derivatives Masks

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$$M_x = \frac{1}{4} \begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix} \quad M_y = \frac{1}{4} \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix} \quad M_t = \frac{1}{4} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

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$$I_x = M_x * (I_1 + I_2) \quad I_y = M_y * (I_1 + I_2) \quad I_t = M_t * (I_2 - I_1)$$

Coupled Gauss-Seidel relaxation

Horn-Schunck Algorithm

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Coupled Gauss-Seidel relaxation

$$u^{k+1} = \bar{u}^k - I_x \frac{I_x \bar{u}^k + I_y \bar{v}^k + I_t}{\alpha + I_x^2 + I_y^2}$$
$$v^{k+1} = \bar{v}^k - I_y \frac{I_x \bar{u}^k + I_y \bar{v}^k + I_t}{\alpha + I_x^2 + I_y^2}$$

Multigrid Scheme

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- ✘ Discretization coarse grid approximation (DCA approach)

A Simple Illustration

- Intensity value :

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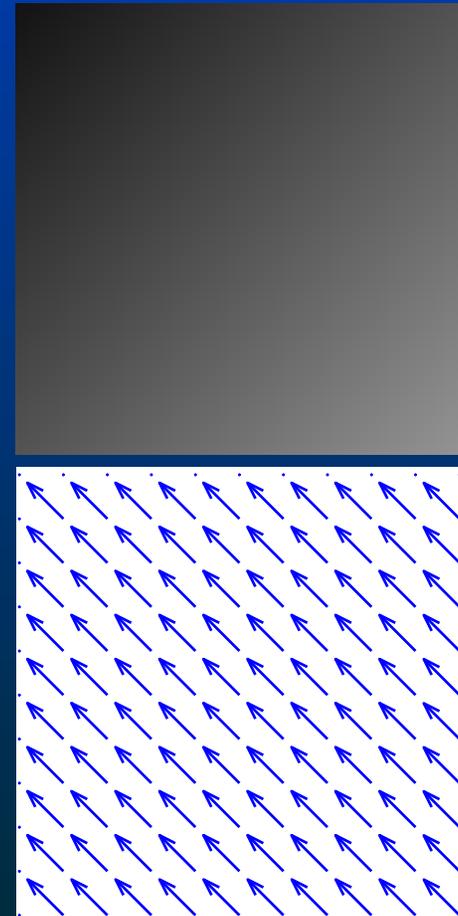
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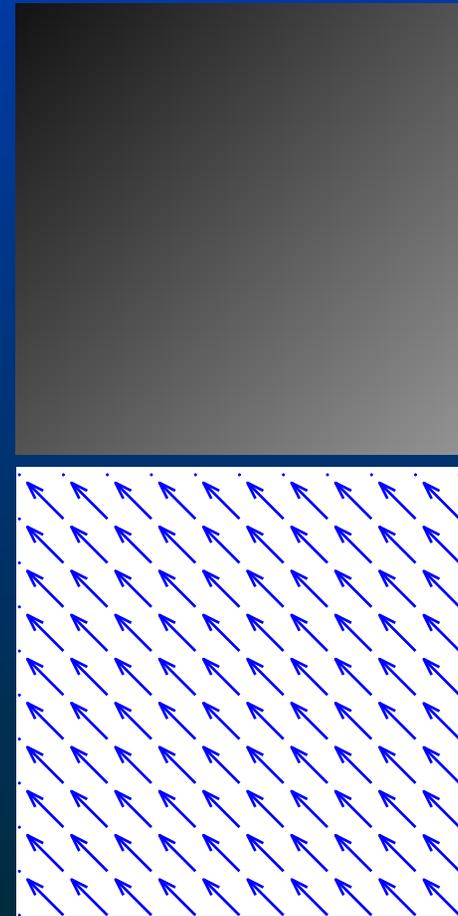
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$\alpha = 1$ and $u_0 \neq v_0$	
Method	Convergence rate
Horn-Schunck	0.998
V(1,0)	0.370
V(1,1)	0.183
V(2,1)	0.116
V(3,3)	0.056



Variational Multigrid

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$$f = -I_x I_t \quad , \quad g = -I_y I_t$$

$$L = L_d + L_r \quad , \quad L_r = \begin{pmatrix} -\alpha\Delta & 0 \\ 0 & -\alpha\Delta \end{pmatrix}$$

and L_d is a 2x2 block diagonal matrix with entries $\begin{pmatrix} I_x^2 & I_x I_y \\ I_x I_y & I_y^2 \end{pmatrix}$

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L_d (positive semi-definite)

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L_d (positive semi-definite) + L_r (positive definite)

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L_d (positive semi-definite) + L_r (positive definite) = L (positive definite).

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$$Lw = F \quad \iff \quad w = \arg \min_{\Omega} a(z)$$

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Galerkin Approach

Let $I_H^h : \Omega^H \mapsto \Omega^h$ be a full rank linear mapping.

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Let $I_H^h : \Omega^H \mapsto \Omega^h$ be a full rank linear mapping.

An optimal coarse grid correction $I_H^h w_H$ of w_h is characterized by

$$((I_H^h)^T L_h I_H^h) w_H = (I_H^h)^T (F - L_h w_h)$$

Variational Multigrid

The CGO is chosen then as follows

$$L_H = I_h^H L_h I_H^h \quad \text{and} \quad I_h^H = (I_H^h)^T$$

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For our system, we get

$$\begin{aligned} L_H &= \begin{pmatrix} R & 0 \\ 0 & R \end{pmatrix} \begin{pmatrix} L_h^1 & L_h^2 \\ L_h^2 & L_h^3 \end{pmatrix} \begin{pmatrix} P & 0 \\ 0 & P \end{pmatrix} \\ &= \begin{pmatrix} RL_h^1 P & RL_h^2 P \\ RL_h^2 P & RL_h^3 P \end{pmatrix} \end{aligned}$$

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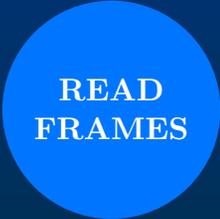
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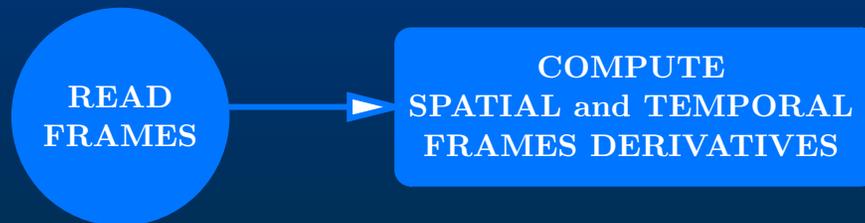
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FRAMES

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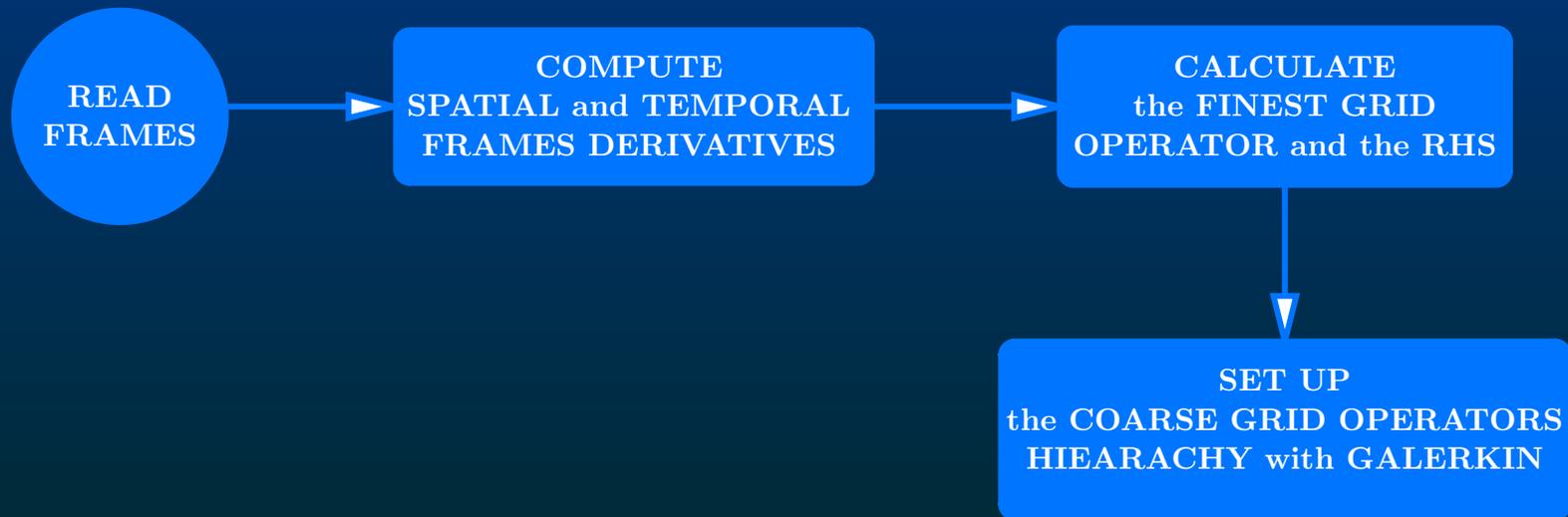


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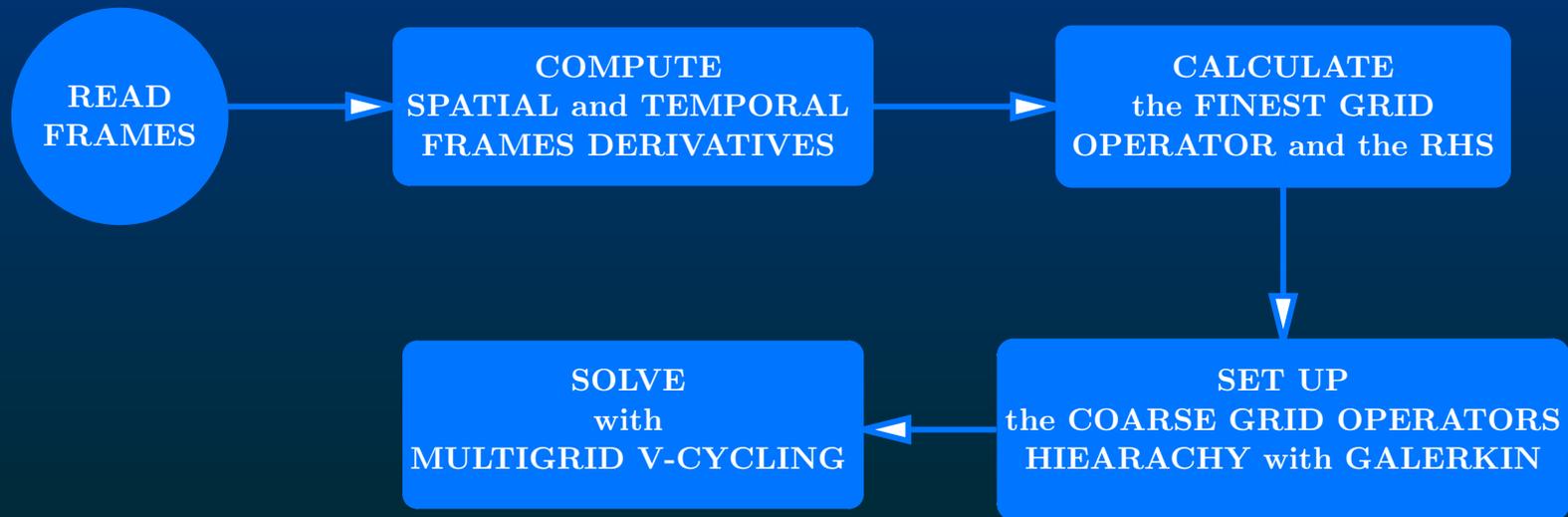


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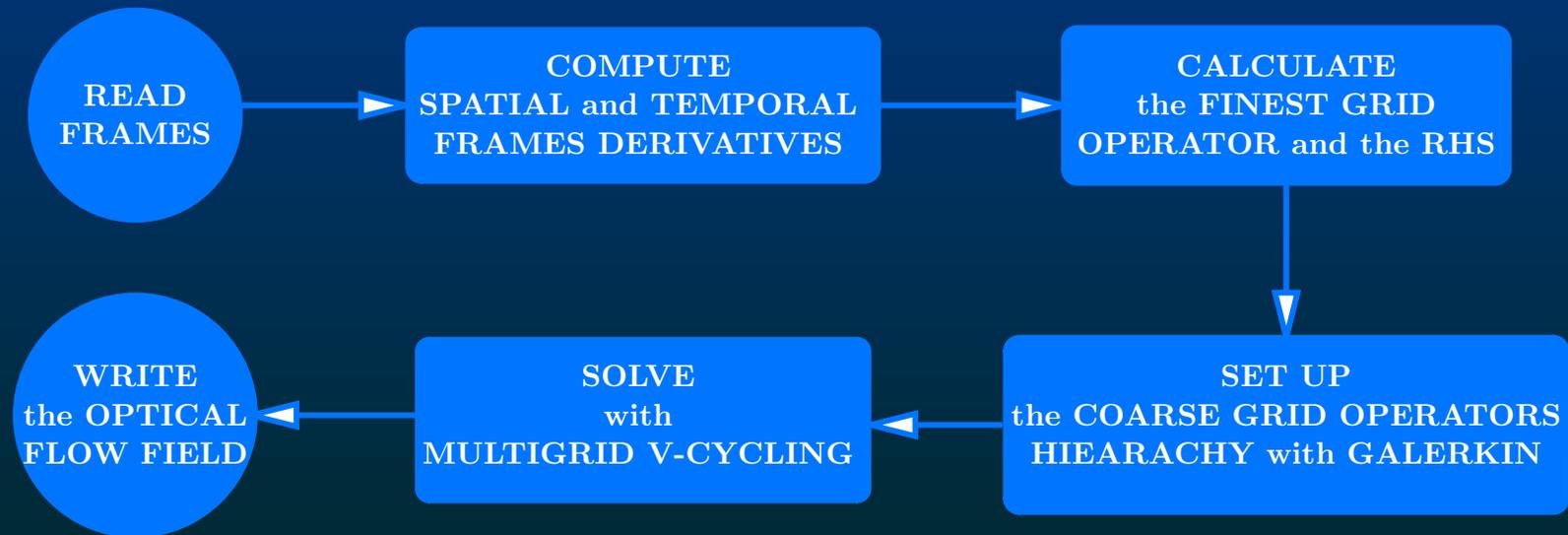


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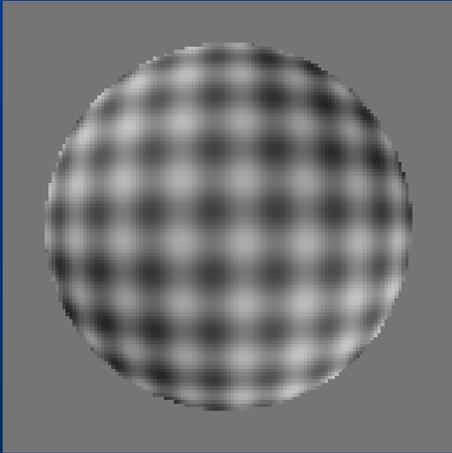
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Experimental Results

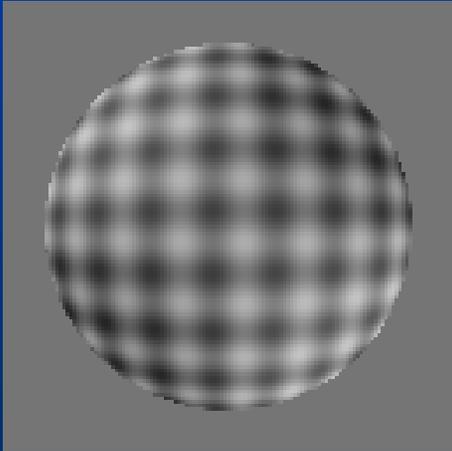


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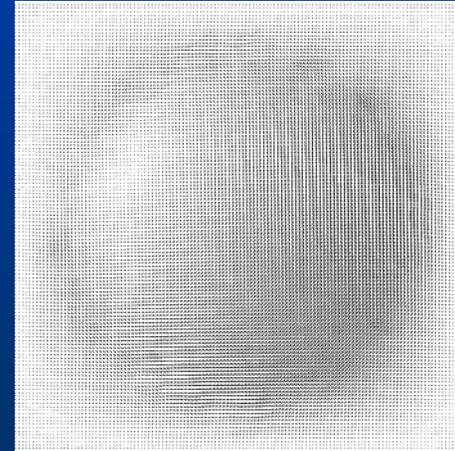


A rotating sphere (128x128)

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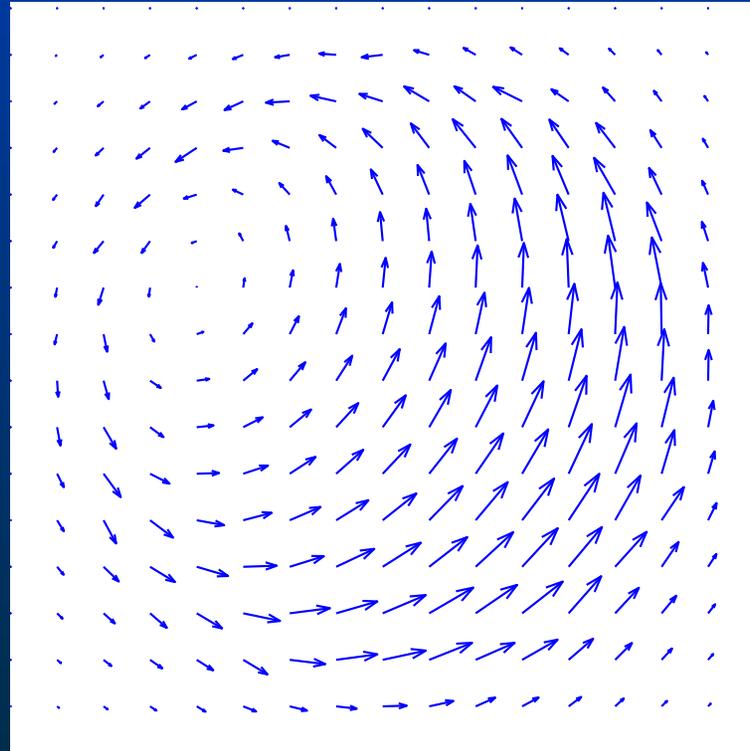


A rotating sphere (128x128)



the flow field

Experimental Results



the flow field after scaling

Experimental Results



The marble sequence (512x512)

Experimental Results

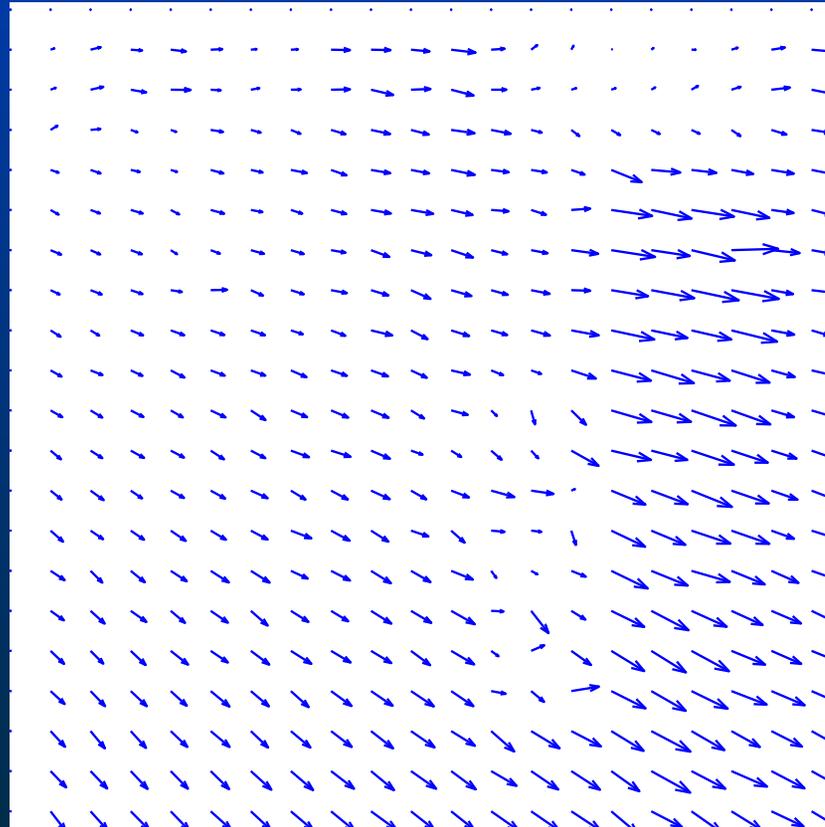


The marble sequence (512x512)



the flow field

Experimental Results



the flow field after scaling

Experimental Results

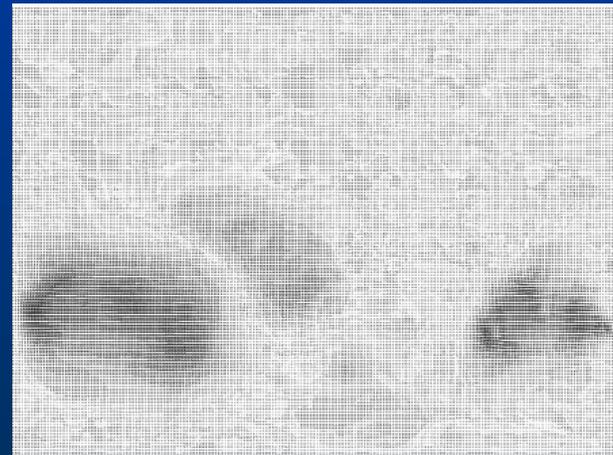


The Hamburg taxi sequence (256x190)

Experimental Results

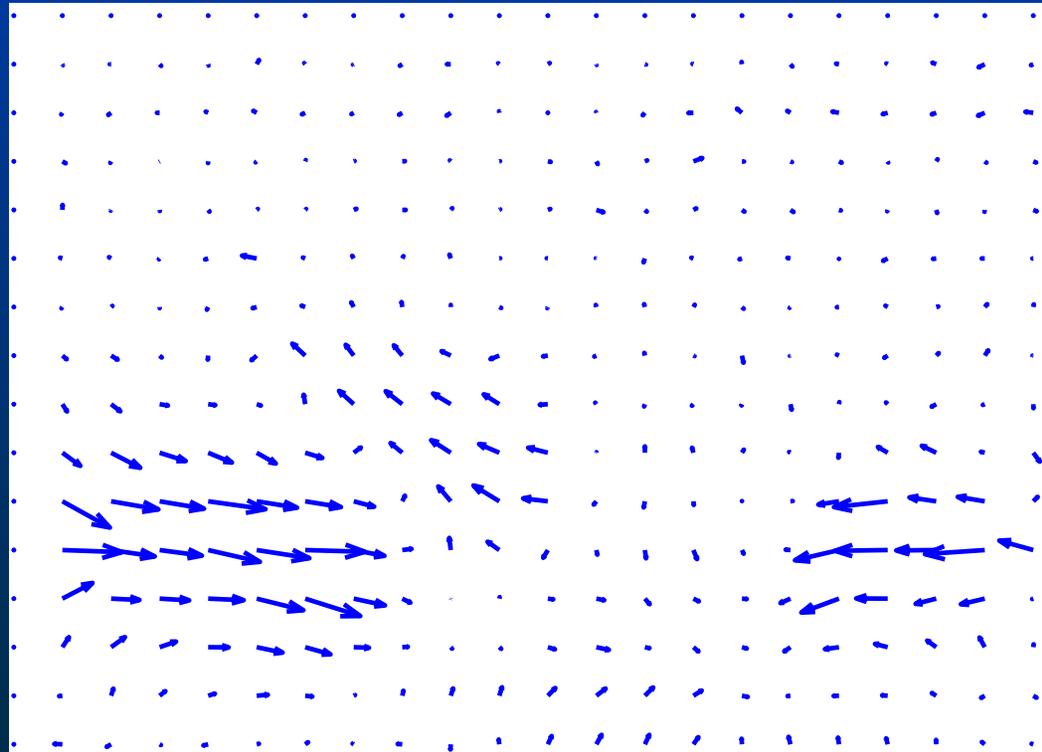


The Hamburg taxi sequence (256x190)



the flow field

Experimental Results



the flow field after scaling

Experimental Results

$\alpha = 5$ and $u_0 = v_0 = 0$ on P4 2.4GHz						
	Sphere		Marble		Taxi	
	ρ	CPU	ρ	CPU	ρ	CPU
Horn-Schunck	0.98	3.9	0.98	106	0.98	20.8
VMG V(2,1)	0.15	1.1	0.43	23.8	0.45	4.3

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- * need matrix dependent transfer grid operators
- Galerkin leads to high memory costs
⇒ search for better representation of the CGOs

Future work

- Application in medicine imaging
- Consider other regularization (e.g. by Nagel, Weickert ...)

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PLEASE WAKE UP!

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THANKS FOR YOUR ATTENTION