AMG Algorithm for Helmholtz Eigenvalue Problem (1D)

Ira Livshits

University of Central Arkansas

April 1, 2003

11th Copper Mountain Multigrid Conference

Wave Ray Algorithm (Brandt, Livshits (1997))

Helmholtz Equation:

$$Lu(x) = \Delta u(x) + k^2 u(x) = f(x)$$

Problematic Error:

$$e(x) = a_1(x)e^{ikx} + a_2(x)e^{-ikx}$$

Corresponded Residual:

$$r(x) = r_1(x)e^{ikx} + r_2(x)e^{-ikx}$$

Ray Residual Approximation:

$$I(r(x)e^{-ikx}) = I(r_1(x) + r_2(x)e^{-2ikx}) \approx r_1(x)$$

Limitations

In

$$r(x) = r_1(x)u_1(x) + r_2(x)u_2(x)$$

need to know two solutions of homogeneous Helmholtz equations:

$$Lu_i \approx 0$$

Eigenvalue Problem with Periodic B.C.

$$Lu(x) = \Delta u(x) + k^2(x)u(x) = \lambda u(x), \qquad x \in \Omega$$

$k(x)$ is periodic in Ω.

Ray Approach

Goal: Find eigenvalues $\lambda_1 \approx \lambda_2$ and their eigenvectors u_1 and u_2

Given: Approximations $u_1^\star$ and $u_2^\star$

Ray Representation:

$$u_1(x) = u_1^\star(x)v_{1,1}(x) + u_2^\star(x)v_{1,2}(x)$$

and

$$u_2(x) = u_1^\star(x)v_{2,1}(x) + u_2^\star(x)v_{2,2}(x)$$

where $v_{i,j}$ are *smooth ray* vectors.

Starting Point

$$Lu_1^n = \lambda_1 u_1^n$$

n is a fine wave level

Ray Representation

$$Lu_1^n = \lambda_1 u_1^n$$

$$Lu_1^n v_{1,1}^n + Lu_2^n v_{1,2}^n = \lambda_1 (u_1^n v_{1,1}^n + u_2^n v_{1,2}^n)$$

Interpolation Operators

$$Lu_1^n v_{1,1}^n + Lu_2^n v_{1,2}^n = \lambda_1 (u_1^n v_{1,1}^n + u_2^n v_{1,2}^n)$$

$$LI_1 v_{1,1}^{n-1} + LI_2 v_{1,2}^{n-1} = \lambda_1 (I_1 v_{1,1}^{n-1} + I_2 v_{1,2}^{n-1})$$

where

$$u_1^n v_{1,1}^n \leftarrow I_1 v_{1,1}^{n-1}, \quad u_1 = I_1(\mathbf{1})$$

$$u_2^n v_{1,2}^n \leftarrow I_2 v_{1,2}^{n-1}, \quad u_2 = I_2(\mathbf{1})$$

Restriction to the Coarser Grid

$$LI_1 v_{1,1}^{n-1} + LI_2 v_{1,2}^{n-1} = \lambda_1 (I_1 v_{1,1}^{n-1} + I_2 v_{1,2}^{n-1})$$

$$(I_1)^t LI_1 v_{1,1}^{n-1} + (I_1)^t LI_2 v_{1,2}^{n-1} = \lambda_1 ((I_1)^t I_1 v_{1,1}^{n-1} + (I_1)^t I_2 v_{1,2}^{n-1})$$

$$(I_2)^t LI_1 v_{1,1}^{n-1} + (I_2)^t LI_2 v_{1,2}^{n-1} = \lambda_1 ((I_2)^t I_1 v_{1,1}^{n-1} + (I_2)^t I_2 v_{1,2}^{n-1})$$

Ray Operators

$$(I_1)^t L I_1 v_{1,1}^{n-1} + (I_1)^t L I_2 v_{1,2}^{n-1} = \lambda_1((I_1)^t I_1 v_{1,1}^{n-1} + (I_1)^t I_2 v_{1,2}^{n-1})$$

$$(I_2)^t L I_1 v_{1,1}^{n-1} + (I_2)^t L I_2 v_{1,2}^{n-1} = \lambda_1((I_2)^t I_1 v_{1,1}^{n-1} + (I_2)^t I_2 v_{1,2}^{n-1})$$

$$A_{1,1} v_{1,1}^{n-1} + A_{1,2} v_{1,2}^{n-1} = \lambda_1(B_{1,1} v_{1,1}^{n-1} + B_{1,2} v_{1,2}^{n-1})$$

$$A_{2,1} v_{1,1}^{n-1} + A_{2,2} v_{1,2}^{n-1} = \lambda_1(B_{2,1} v_{1,1}^{n-1} + B_{2,2} v_{1,2}^{n-1})$$

Finest Ray Operators

$$\mathbf{A} = \begin{bmatrix} A_{1,1} & A_{1,2} \\ A_{2,1} & A_{2,2} \end{bmatrix}.$$

$$A_{i,j} = (I_i)^t L I_j, \qquad u_j = I_j(\mathbf{1})$$

$$\mathbf{B} = \begin{bmatrix} B_{1,1} & B_{1,2} \\ B_{2,1} & B_{2,2} \end{bmatrix}.$$

$$B_{i,j} = (I_i)^t I_j$$

Coarse Ray Operators

$$\mathbf{A^k} = \begin{bmatrix} A^k_{1,1} & A^k_{1,2} \\ A^k_{2,1} & A^k_{2,2} \end{bmatrix}.$$

$$A^k_{i,j} = (I)^t A^{k+1}_{i,j} I$$

$$\mathbf{B^k} = \begin{bmatrix} B^k_{1,1} & B^k_{1,2} \\ B^k_{2,1} & B^k_{2,2} \end{bmatrix}.$$

$$B^k_{i,j} = (I)^t B^{k+1}_{i,j} I$$

I is a linear interpolation operator

Ray Formulation

Ray Vectors:

$$\mathbf{V_i^k} = \begin{pmatrix} v_{i,1}^k \\ v_{i,2}^k \end{pmatrix}$$

Ray Operators:

$$\mathbf{A^k}, \mathbf{B^k}$$

Ray Equations:

$$\mathbf{A^k V_i^k} = \lambda_i \mathbf{B^k V_i^k}$$

Algorithm

1. Wave Relaxation: $Lu_i^n = \lambda_i u_i^n, \quad i = 1, 2$

2. Ray Grids: construct operators $\mathbf{A^k}, \mathbf{B^k}, k = n-1, \ldots, 0$

3. Coarsest Ray Grid: Solve for $\mathbf{V_i^0}$ and λ_i, using Rayleigh-Ritz Projection $\mathbf{A^0 V_i^0} = \lambda_i \mathbf{B^0 V_i^0}, \quad i = 1, 2,$
   starting with $v_{1,1} = v_{2,2} = \mathbf{1}, \quad v_{2,1} = v_{1,2} = \mathbf{0}$

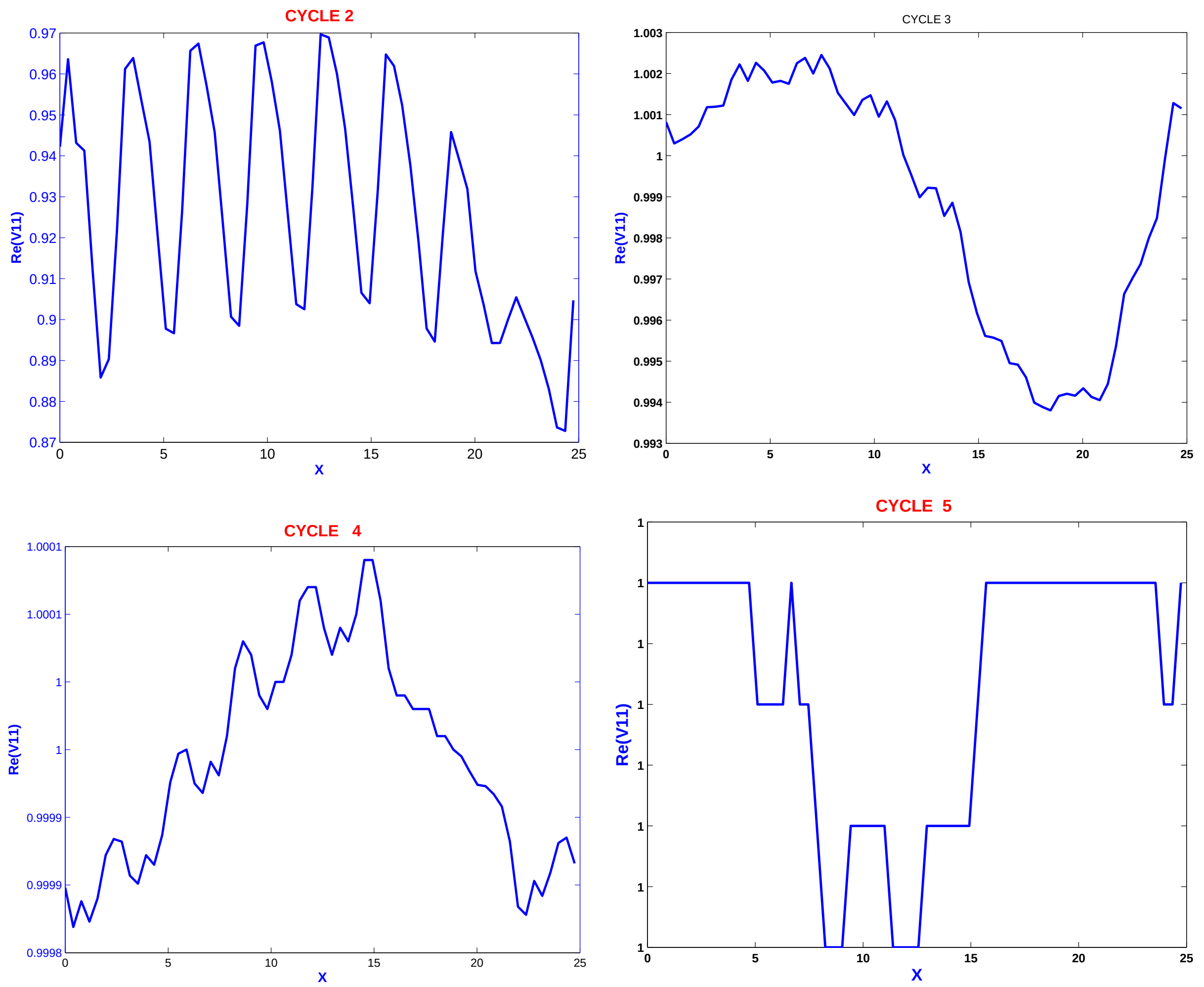
4. Ray Interpolation and Ray Relaxation:
   $\mathbf{V_i^k} = \mathbf{I}(\mathbf{V_i^{k-1}}), \quad \mathbf{A^k V_i^k} = \lambda_i \mathbf{B^k V_i^k}$
   $i = 1, 2, \quad k = 1, \ldots, n-1$

5. Wave Interpolation: $u_i^n = I_1 v_{i,1}^{n-1} + I_2 v_{i,2}^{n-1}, \quad i = 1, 2$

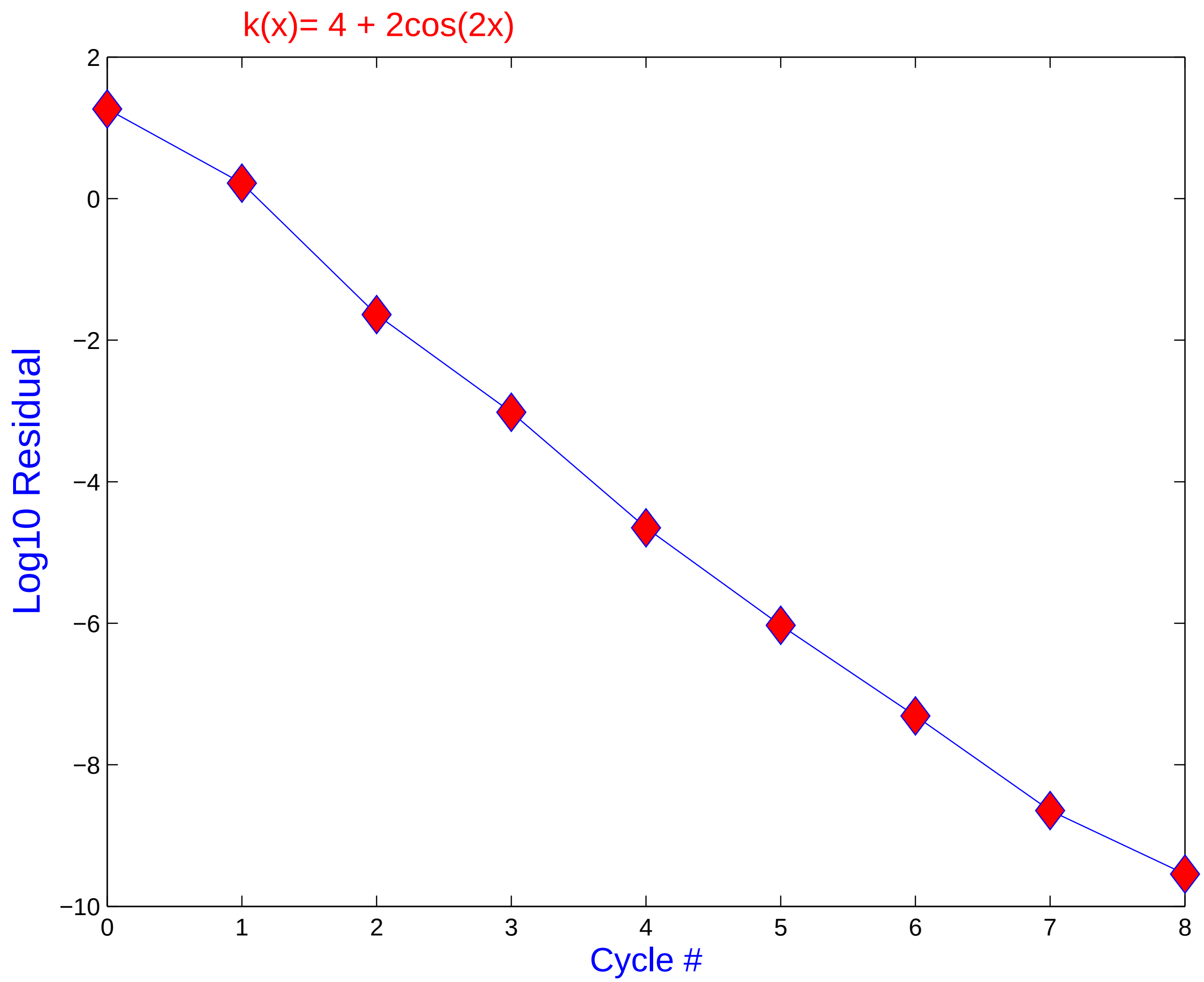
Model Problem

$$u_{xx} + (4 + 2\cos(2x))^2 u = \lambda u(x), \qquad x \in [0 : 8\pi]$$

Smoothness of Ray Functions



Residual Convergence



Technical Summary

- Finest grids with $kh \ll 1$ can be *wave* grids.
- Laplace like amount of work on finest grids.
- Few relaxation sweeps on intermediate ray levels.
- Need good *initial* approximations $u_1^\star$ and $u_2^\star$.
- Interpolation: linear.
- For Cycle 1: coarsest grid matrix $\mathbf{A}$ is almost block diagonal: $A_{1,2} = A_{2,1}^T \approx 0$ (separate systems for $v_{1,1}$ and $v_{1,2}$, and $v_{2,1}$ and $v_{2,2}$). This is due to orthogonality of $u_1^\star$ and $u_2^\star$.

Next...

- Finest grid othogonalization of u_1 and u_2 ? (more important for 2D)
- Correlation between $k(x)$ and *ray* mesh sizes
- Eigenbasis (A.Brandt's Talk)
- Two-dimensional Helmholtz Operators