Accurate multigrid techniques for computing singular solutions of elliptic problems

Lehrstuhl für Informatik 10 (Systemsimulation)
Friedrich–Alexander–Universität
Erlangen–Nürnberg

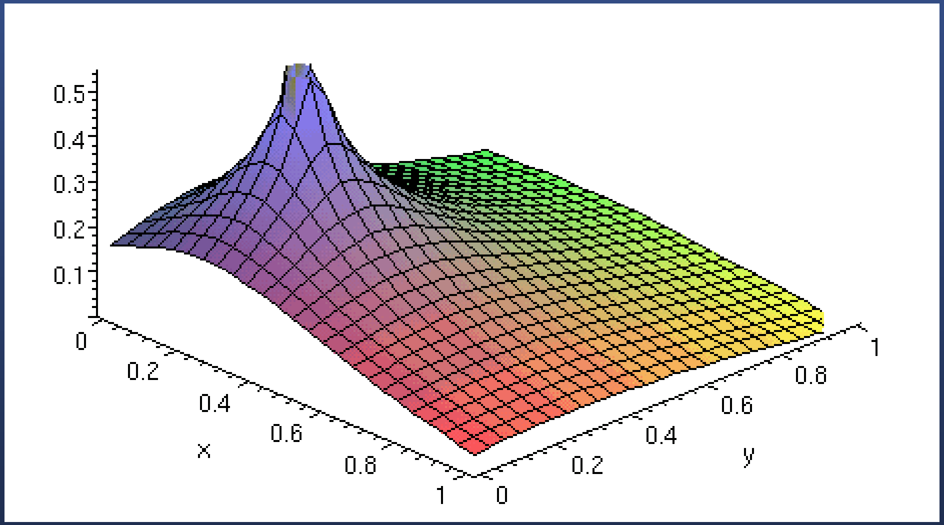
U. Rüde, H. Köstler, M. Kowarschik, M. Mohr

Outline

Problem Description

Discretization of elliptic PDEs with

singularities $\left\{ \begin{array}{l} \text{point load} \\ \text{dipole} \\ \text{quadrupole} \end{array} \right\}$ in the source terms

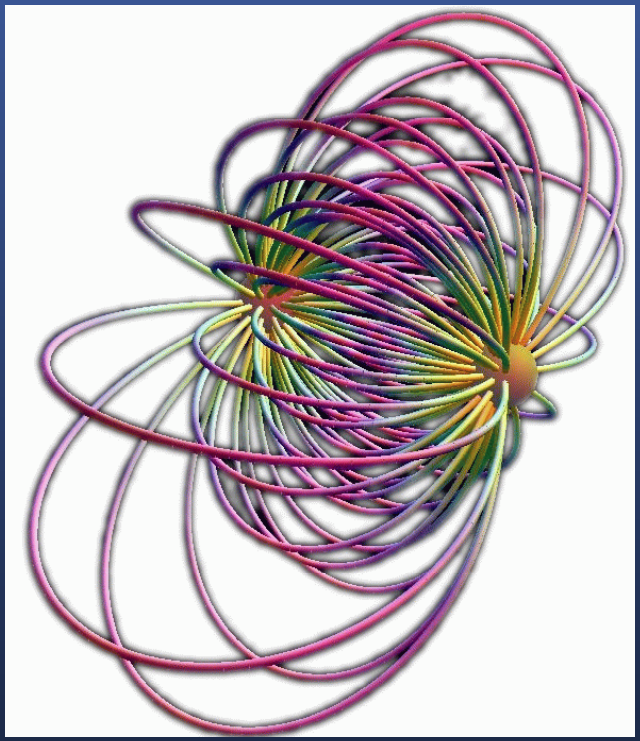


standard discretization relies on smoothness $\Rightarrow$ $\left\{ \begin{array}{l} \bullet \text{ accuracy deteriorates} \\ \bullet \text{ standard analysis fails} \end{array} \right.$

Applications

Fields of application in which problems with singularities of this kind arise are manifold, e.g.

- point sources and sinks in porous media flow that are described by Dirac delta functions
- point loads and dipoles as source terms inducing electrostatic potentials
- . . .



Our Motivation

Reconstruction of electrical behaviour inside the head from measurements (Electroencephalography)

Inverse problem important for Neurology and Neurosurgery

Neurosurgery at Erlangen University Head Center

Localization of an epileptic focus

Zenger Correction

Idea: Replace the original generalized function by its numerical equivalent on a uniform grid.

1. no local refinement

2. Construction of numerical representation

   - analytical integration of delta function yields smooth function

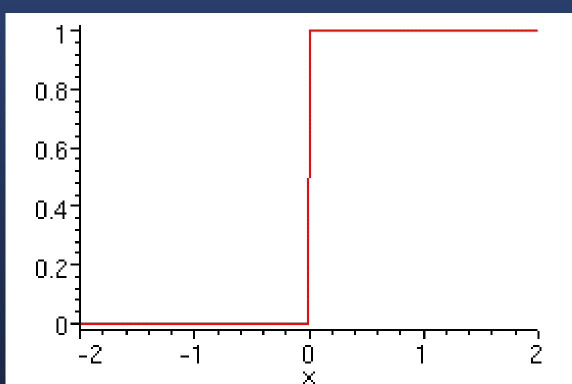
   - numerical differentiation yields h-dependent discrete equivalent of delta function

Generalized Functions

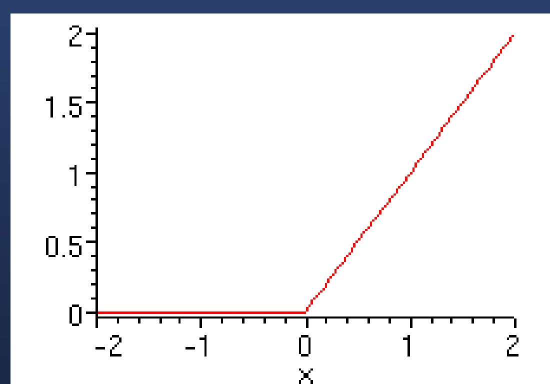
Definition: The family of functions $H_i : \mathbb{R} \to \mathbb{R}$, $i \in \mathbb{Z}$ is defined by

$$H_0(x) := \begin{cases} 0 & : \quad x \leq 0 \\ 1 & : \quad x > 0 \end{cases}, \qquad H_i(x) := \begin{cases} H'_{i-1}(x) & : \quad i > 0 \\ \int\limits_{-\infty}^{x} H_{i+1}(\xi) d\xi & : \quad i < 0 \end{cases}$$

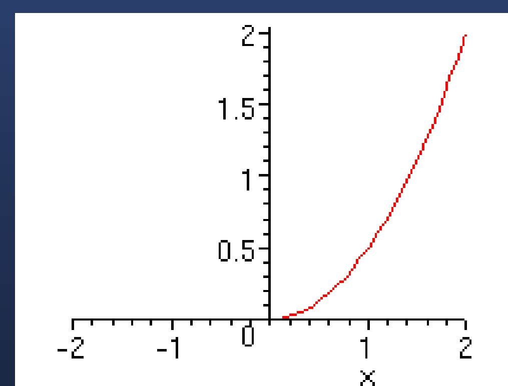
where H_0 is the Heaviside step function and H_1 the Dirac-δ-function.



H_0

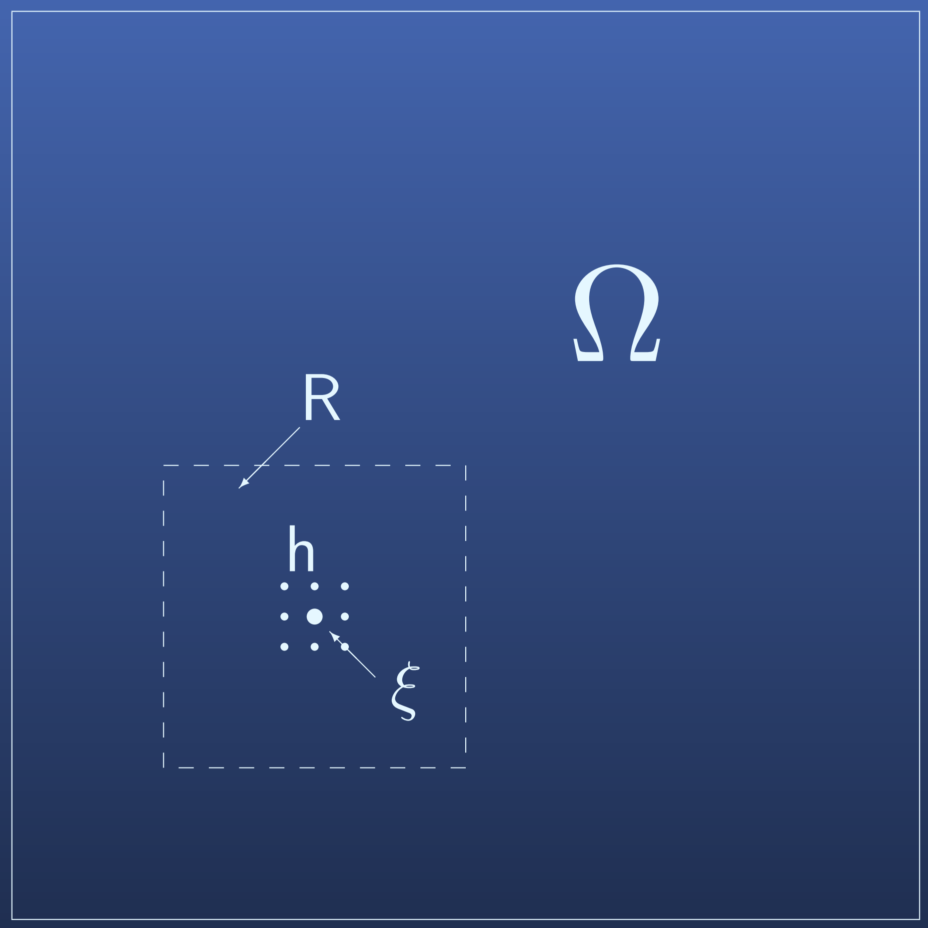


H_{-1}



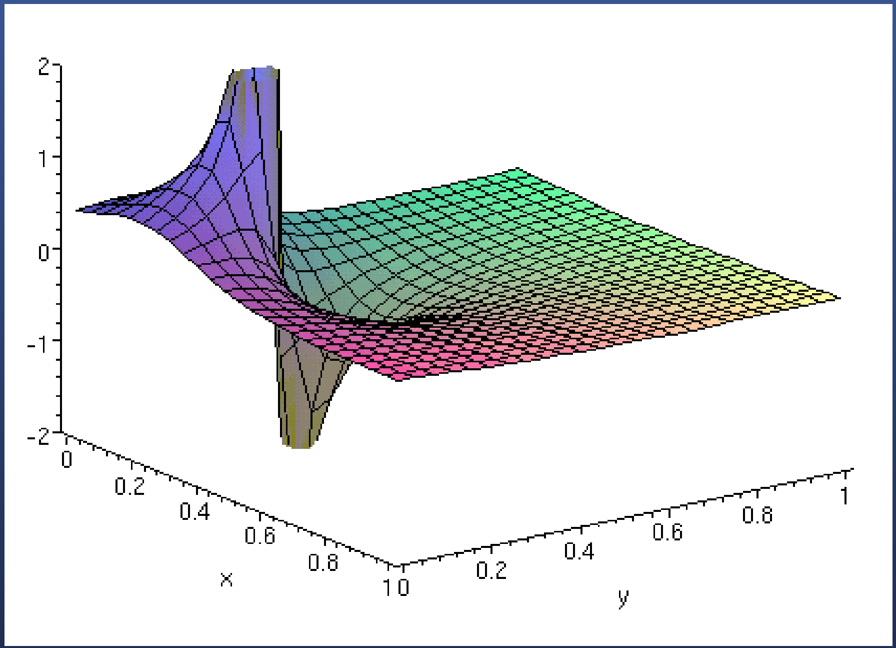
H_{-2}

Zenger Correction in the Unit Square

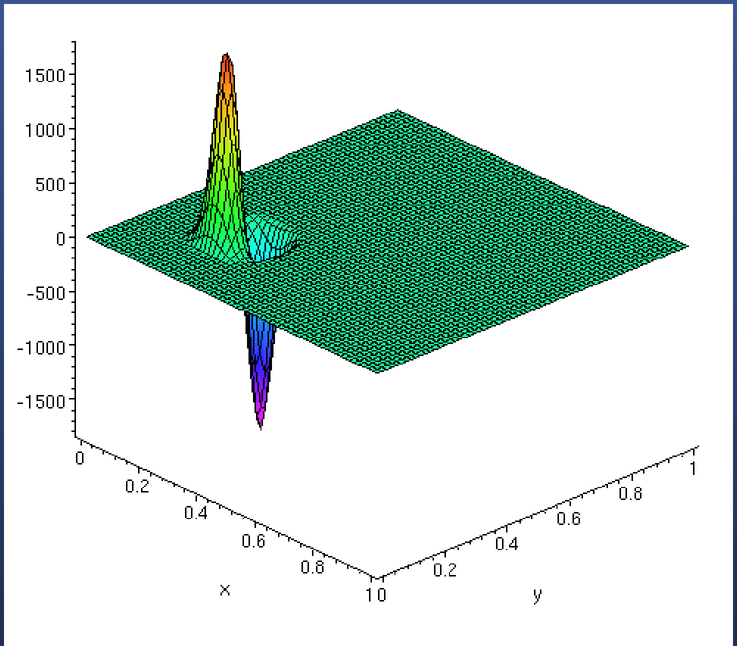


Analytic Solution for a Dipole

$$u_d^*(x,y) \;=\; \vec{v}\,\nabla\, u_p^*(x,y) \;=\; \frac{v_1(x-x_0)+v_2(y-y_0)}{2\pi r^2}$$



Analytic Solution



Right hand side with Zenger correction

Zenger Correction: Summary

Advantages

+ analytical solution is not needed for correction
+ no changes to grid or solver
+ correction is been applied to a fixed number of points ($\Rightarrow$ independent of mesh-size)
+ the pollution effect is eliminated

Disadvantages

– the accuracy breaks down near the singularity

Richardson Extrapolation

- based on the asymptotic error expansion

$$u_h - u^* = h^2 e_2 + h^4 e_4 + \ldots + \mathcal{O}(h^{2k})$$

- we use a fine grid with mesh-size h and a coarse grid with $H = 2h$
- on the coarse grid a higher accuracy is achieved through the combined solution

$$\widetilde{u}_H = \frac{4}{3}\,\widehat{I}_h^H\,u_h - \frac{1}{3}\,u_H$$

  $\Rightarrow$ so the final accuracy can be improved from $\mathcal{O}(h^2)$ to $\mathcal{O}(h^4)$

- u_h and u_H are naturally computed with full multigrid

τ-Extrapolation

- τ-Extrapolation is a multigrid specific technique and works for both CS and FAS
- For CS the defects of two different grid levels are combined
- Higher accuracy is achieved by modifying the correction at the finest grid level only

$$\widetilde{u}_h^{m+1} = u_h^m + I_H^h A_H^{-1} \left(\frac{4}{3} I_h^H (f_h - A_h u_h^m) - \frac{1}{3} (I_h^H f_h - A_H \widehat{I}_h^H u_h^m) \right)$$

- special care has to be taken when choosing the restriction operator and the smoothing procedure in order not to destroy the high accuracy

Numerical Experiments

- model problem: Poisson equation

$$\begin{aligned} \Delta u &= f \quad \text{in } \Omega = (0,1)^3 \\ u &= g \quad \text{on } \partial\Omega \end{aligned}$$

- finite differences, equidistant grid, mesh-size h.

- correction scheme multigrid solver

$$f = \left\{ \begin{array}{r} \text{point load} \\ \text{dipole} \\ \text{quadrupole} \end{array} \right\} \text{ at } \xi = (0.26, 0.26, 0.26) \text{ not a grid point}$$

Discretization Errors for a Dipole

h	$(\frac{1}{2},\frac{1}{2},\frac{1}{2})$	L_∞ Ω	L_1 Ω	L_2 Ω	L_1 $\Omega \setminus R$	L_2 $\Omega \setminus R$
$\frac{1}{8}$	2.04e-01	2.85e+02	9.37e-01	1.54e+01	1.01e-01	1.24e-01
$\frac{1}{16}$	5.29e-02 (3.9)	2.75e+02	1.12e-01 (8.3)	4.73e+00 (3.2)	2.27e-02 (4.5)	2.96e-02 (4.2)
$\frac{1}{32}$	1.34e-02 (4.0)	2.16e+02	1.83e-02 (6.1)	1.26e+00 (3.7)	5.19e-03 (4.4)	7.10e-03 (4.2)
$\frac{1}{64}$	3.36e-03 (4.0)	6.13e+02	7.25e-03 (2.5)	1.26e+00 (1.0)	1.23e-03 (4.2)	1.73e-03 (4.1)
$\frac{1}{128}$	8.39e-04 (4.0)	4.92e+03	4.31e-03 (1.7)	3.45e+00 (0.4)	3.00e-04 (4.1)	4.28e-04 (4.1)
$\frac{1}{256}$	2.10e-04 (4.0)	4.93e+03	1.44e-03 (3.0)	1.42e+00 (2.4)	7.39e-05 (4.1)	1.06e-04 (4.0)

$\Omega =$ problem domain, $\quad R =$ region around singularity

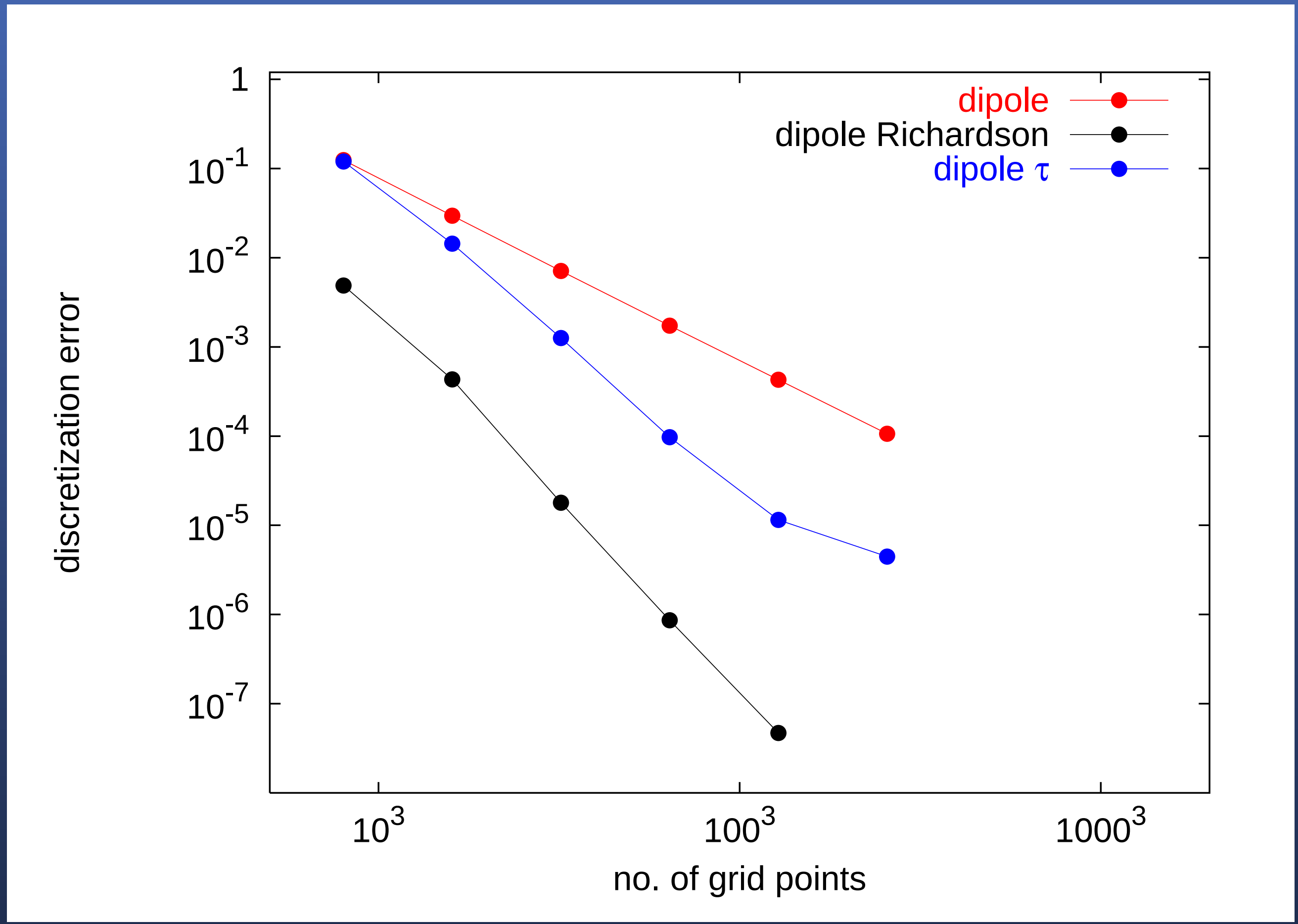
Discretization Errors for a Dipole and Richardson Extrapolation

h	$(\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$	L_∞ Ω	L_1 Ω	L_2 Ω	L_1 $\Omega \setminus R$	L_2 $\Omega \setminus R$
$\frac{1}{8}$	1.69e-03	2.79e+02	8.25e-01	1.51e+01	1.73e-03	4.88e-03
$\frac{1}{16}$	1.25e-04 (13.5)	2.37e+02	7.52e-02 (11.0)	4.09e+00 (3.7)	1.02e-04 (17.0)	4.32e-04 (11.3)
$\frac{1}{32}$	8.14e-06 (15.4)	6.46e+01	4.70e-03 (16.0)	3.89e-01 (10.5)	4.67e-06 (21.8)	1.79e-05 (24.2)
$\frac{1}{64}$	5.10e-07 (16.0)	7.86e+01	1.58e-03 (3.0)	2.21e-01 (1.8)	2.45e-07 (19.0)	8.63e-07 (20.7)
$\frac{1}{128}$	3.20e-08 (16.0)	2.30e+03	1.69e-03 (0.9)	1.62e+00 (0.1)	1.40e-08 (17.5)	4.70e-08 (18.4)

Discretization Errors for a Dipole and τ-Extrapolation

h	$(\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$	L_∞ Ω	L_1 Ω	L_2 Ω	L_1 $\Omega \setminus R$	L_2 $\Omega \setminus R$
$\frac{1}{8}$	1.52e-01	2.86e+02	9.67e-01	1.54e+01	9.64e-02	1.20e-01
	14.7		8.4	3.2	9.2	8.3
$\frac{1}{16}$	1.03e-02	2.83e+02	1.15e-01	4.88e+00	1.05e-02	1.44e-02
	12.5		5.9	3.1	12.3	11.4
$\frac{1}{32}$	8.24e-04	2.61e+02	1.97e-02	1.57e+00	8.47e-04	1.26e-03
	128.5		2.3	0.9	13.7	13.0
$\frac{1}{64}$	6.42e-06	7.99e+02	8.43e-03	1.72e+00	6.16e-05	9.71e-05
	0.8		1.7	0.4	9.5	8.5
$\frac{1}{128}$	7.99e-06	5.64e+03	5.03e-03	4.02e+00	6.47e-06	1.15e-05
	17.0		2.7	1.6	3.2	2.6
$\frac{1}{256}$	4.70e-07	7.79e+03	1.84e-03	2.47e+00	2.02e-06	4.45e-06

Extrapolation for increased Accuracy



τ-Extrapolation does not work with $\mathcal{O}(h^4)$ yet (higher order B-Splines needed)!

Discretization Errors for a Quadrupole

h	$(\frac{1}{2},\frac{1}{2},\frac{1}{2})$	L_∞ Ω	L_1 Ω	L_2 Ω	L_1 $\Omega \setminus R$	L_2 $\Omega \setminus R$
$\frac{1}{8}$	9.06e-02	8.01e+01	9.31e-01	6.45e+00	2.03e-01	4.30e-01
	3.6		1.1	0.4	6.1	6.5
$\frac{1}{16}$	2.53e-02	8.06e+02	8.11e-01	1.75e+01	3.32e-02	6.57e-02
	3.0		1.4	0.5	5.1	5.8
$\frac{1}{32}$	8.48e-03	5.64e+03	5.92e-01	3.84e+01	6.53e-03	1.14e-02
	3.9		1.2	0.4	4.4	4.7
$\frac{1}{64}$	2.18e-03	3.76e+04	4.90e-01	9.11e+01	1.47e-03	2.44e-03
	4.4		0.8	0.3	4.2	4.2
$\frac{1}{128}$	4.92e-04	4.06e+05	6.14e-01	3.31e+02	3.51e-04	5.80e-04
	3.4		1.4	0.6	4.1	4.2
$\frac{1}{256}$	1.44e-04	1.33e+06	4.30e-01	5.13e+02	8.50e-05	1.37e-04

Ω = problem domain, R = region around singularity

Bioelectric Field Problem

Governing equation:

$$-\nabla \cdot (\sigma \nabla \Phi) = I$$

σ: conductivity tensor

Φ: Potential field (forward problem)

I: Source terms (inverse problem)

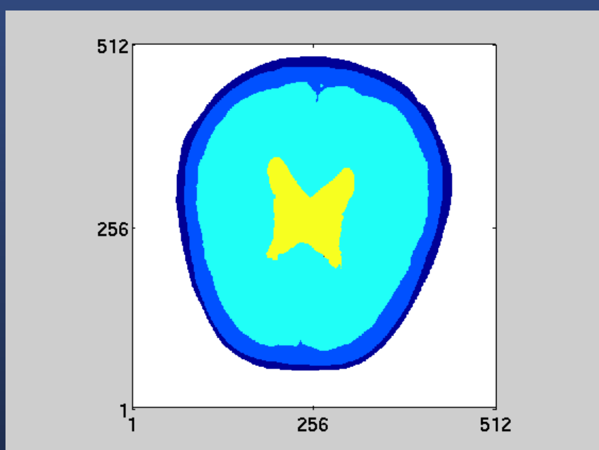
Electrically active brain areas are physically modeled as dipoles

$\Rightarrow$ suitable numerical dipole models are needed

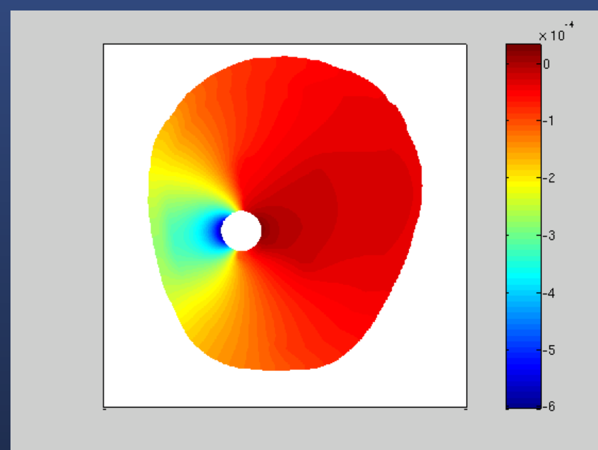
Comparison for a Realistic Setting

We compare the new dipole model to a standard Finite-Volume-Discretization and solve the foward EEG problem on a 2D head slice

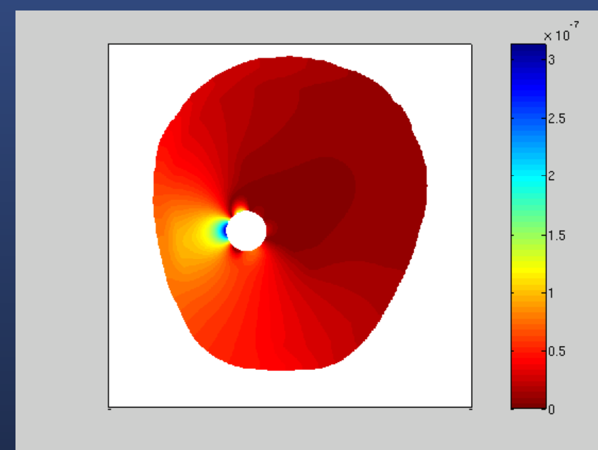
(We use a unit dipole left of the ventricular system, orientation $190°$)



2D head slice



Dipole (simple FVM)



Difference in models

Conclusions & Future Work

- The results show, that the Zenger correction can be
  - extended to 3D
  - extended to dipoles & quadrupoles
  - used in conjunction with extrapolation techniques
- Future work
  - Fix (technical) problems with τ-Extrapolation
  - Evaluate importance of Zenger correction for Bioelectric Field problem
  - Generalize existing 2D-theory

References

- U. Rüde, On the Accurate Computation of Singular Solutions of Laplace's and Poisson's Equation, in Multigrid Methods, ed. S. McCormick, 3rd Copper Mountain Conference, 1987
- C. Zenger, H. Gietl, Improved Schemes for the Dirichlet Problem of Poisson's Equation in the Neighbourhood of Corners, in Numerische Mathematik, 1978