Towards Textbook Multigrid Efficiency for Fluid Flows : Stagnation

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Outline

- Motivation and objective
- Elements of Textbook Multigrid Efficiency (TME)
- Stagnation flow model problems
- Isolation and analysis of difficulties
  - Interior relaxation
  - Boundary closure/relaxation
  - Interactions
- Computational results
- Concluding remarks

Motivation

- Textbook Multigrid Efficiency (TME) : Solutions attained in a few (< 10) minimal work units

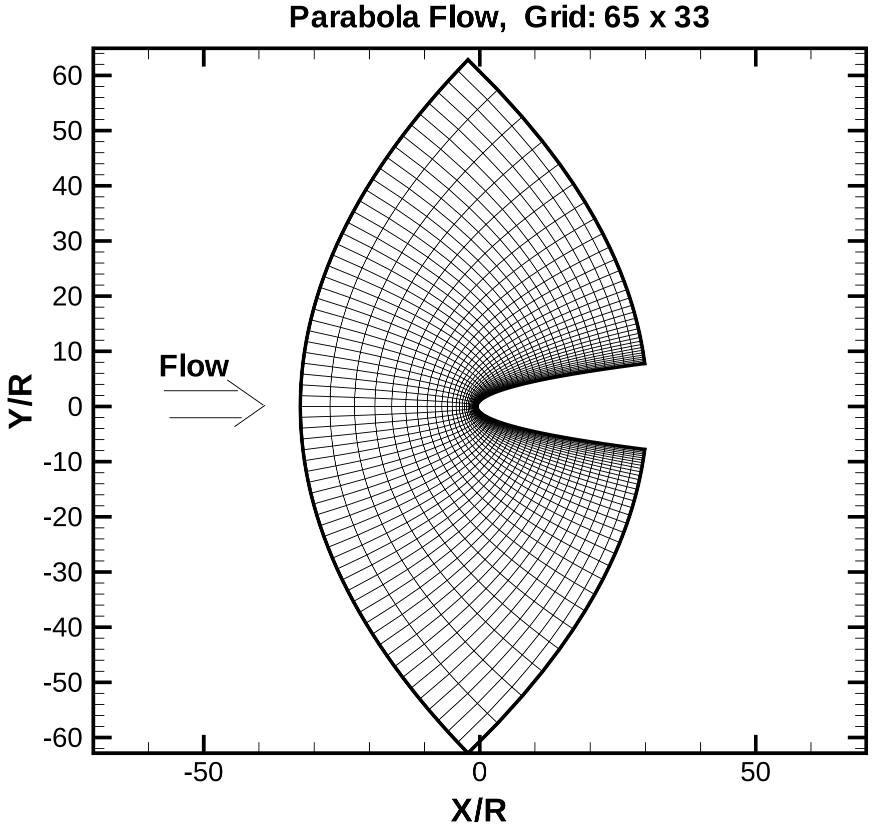
  A **minimal work unit** is the operation count in one target-grid residual evaluation

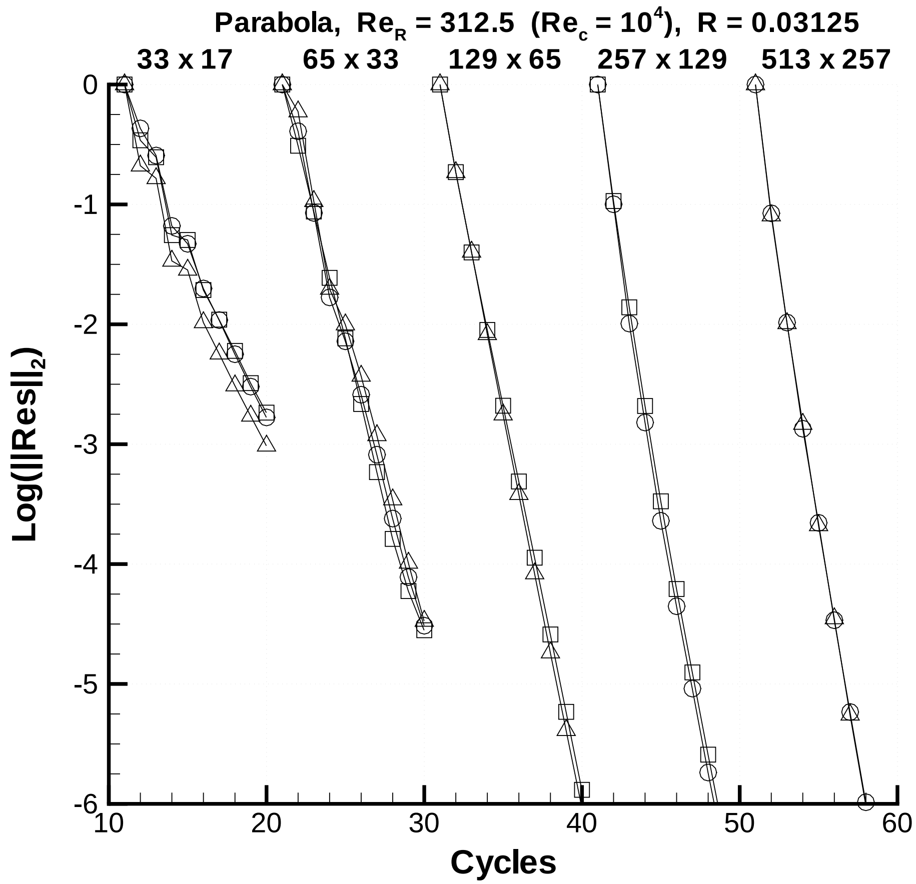
- Motivation : Develop TME algorithms for time-dependent viscous flows applied to flow control applications
  - Builds upon earlier incompressible (Pressure-Poisson) formulations in generalized coordinates
  - Inviscid (Roberts, Sidilkover & Swanson, 1999)
  - Viscous (Swanson, 2001)

Current Objective

- Current Objective : Overcome difficulties encountered in applications associated with stagnation regions
  - Non-optimal asymptotic convergence rates
  - Poor coarse-grid corrections
  - Reduced efficiency for large domain sizes (coarse grids, high stretching)
- Convergence difficulties also experienced in stagnation for preconditioned compresssible formulations (e.g., Turkel, Vatsa & Radespiel, 1996)

Viscous Flow Around a Parabola





Incompressible Navier-Stokes (INS) equations

- Pressure-Poisson Formulation:

$$u\partial_x^u u + v\partial_y^u u - \nu\Delta^h u + \partial_x^c p = 0,$$

$$u\partial_x^u v + v\partial_y^u v - \nu\Delta^h v + \partial_y^c p = 0,$$

$$(\partial_x^c u)^2 + 2(\partial_y^c u)(\partial_x^c v) + (\partial_y^c v)^2 + \Delta^h p = 0.$$

- Efficient relaxation over a *major* part of the domain

$$\mathbf{L}\,\delta\mathbf{q} \;=\; -\mathbf{R}^h(\mathbf{q}^n) \qquad \delta\mathbf{q} = \mathbf{q}^{n+1} - \mathbf{q}^n$$

  – Principal linearization $\mathbf{L}$ is upper triangular (decoupled relaxation)

Principal Linearization
(From the Standpoint of Relaxation)

- The principal linearization $\mathbf{L}$ is derived from the full Newton linearization by removing subprincipal terms

- **Scalar equation:** Retain terms that make major contributions to the residual

- **Systems:** Retain terms that make major contributions to the *determinant* of the matrix operator

Elements of TME : Principal Linearization

(Example - Nonlinear Convection Operator: $u\partial_x^h u$)

Full Linearization: $[u\partial_x^h \ + \ \partial_x^h u]\delta u$

High-Frequency Contribution (local mode analysis):

$$\frac{1}{h}\left[\ u\,\widehat{\partial_x^h}\ +\ h\,\partial_x^h u\,\right]\delta u$$

$$O(u) \quad O(hu_x)$$

Regular flow region ($hu_x < u$) : $L = u\partial_x^h$

Stagnation flow region ($hu_x \approx u$) : $L = u\partial_x^h + (\partial_x^h u)$

Principal Linearization

- Principal Linearization in the **regular flow field**:

$$\mathbf{L} = \begin{bmatrix} Q^u & 0 & 0 \\ 0 & Q^u & 0 \\ 0 & 0 & \Delta^h \end{bmatrix}$$

$$Q^u = u\partial_x^u + v\partial_y^u - \nu\Delta^h$$

$$\det \mathbf{L} = (Q^u)^2\Delta^h$$

- Principal Linearization Near Stagnation (Full Linearization):

$$\mathbf{L} = \begin{bmatrix} Q^u + (\partial_x^u u) & (\partial_y^u u) & \partial_x^c \\ (\partial_x^u v) & Q^u + (\partial_y^u v) & \partial_y^c \\ 2(\partial_x^c u)\partial_x^c + 2(\partial_x^c v)\partial_y^c & 2(\partial_y^c u)\partial_x^c + 2(\partial_y^c v)\partial_y^c & \Delta^h \end{bmatrix}$$

Relaxation Strategy for Factorizable Scheme

- The efficiency goal is the "per relaxation" convergence rates associated with scalar factors in the regular flow field
- Global : decoupled relaxation in the regular flow field
- Local : coupled relaxation near boundaries/singularities
  - The **local** principal linearization is closer to a full linearization and generally not easily analyzed
  - A different discretization might be used
- General approach is to tailor **local** relaxation to **local** properties of the solution

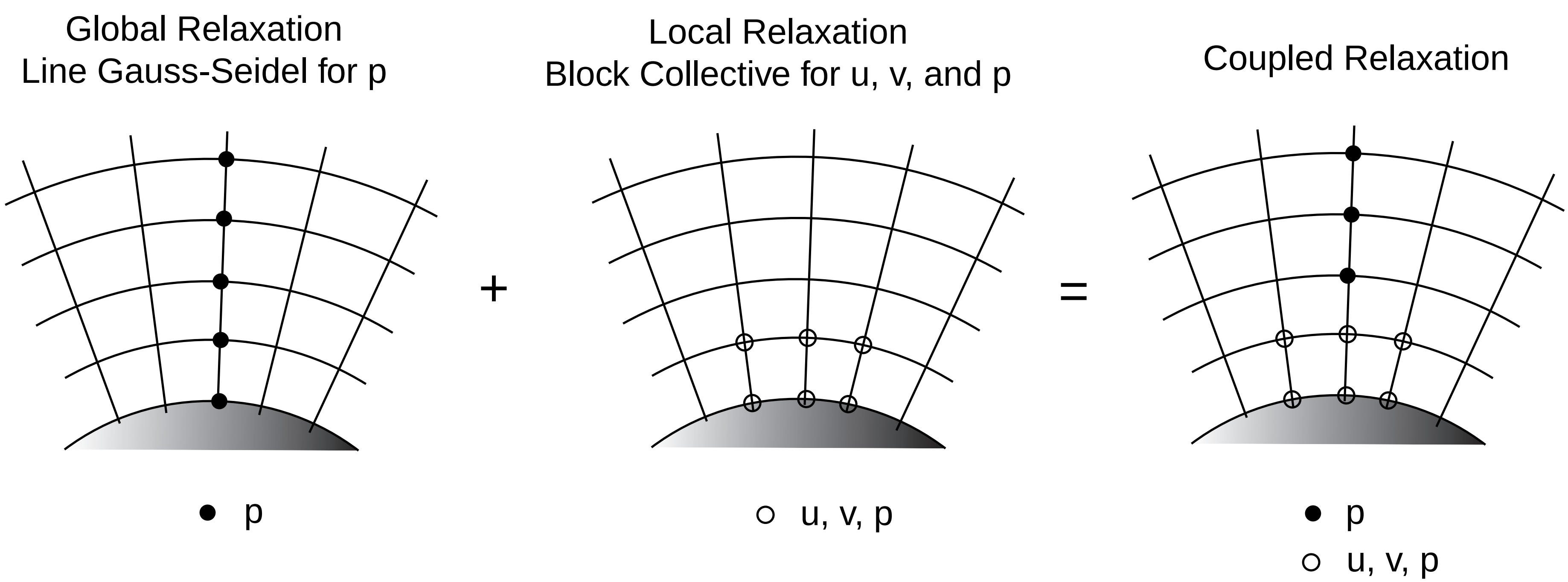
Relaxation Framework

- Linearized equations relaxed at each level

$$\mathbf{L}\delta(\delta\mathbf{q}) = -\mathbf{R}(q) - \frac{\partial \mathbf{R}}{\partial \mathbf{q}}\delta\mathbf{q}$$

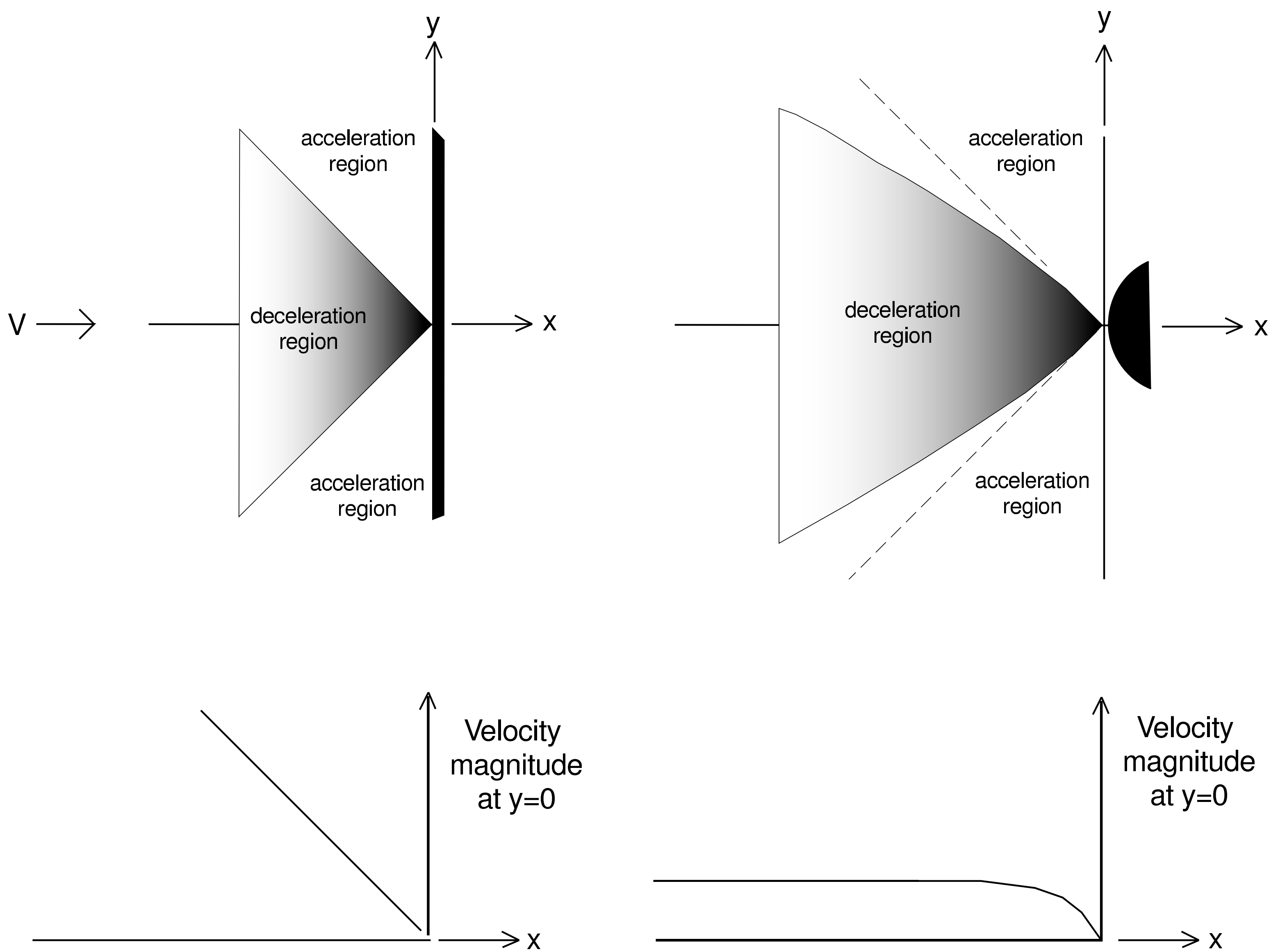
- Global Relaxation : alternating line-implicit
  - Pressure-equation Gauss-Seidel relaxation (velocity is fixed)
  - Relaxation (marching) of momentum equations
- Local Relaxation: equations are solved simultaneously
  - at 5 points near the body surface or outflow
  - at 3 points near the inflow boundary

Coupling of Global and Local Relaxation
(At a Boundary)



Stagnation Flow Model Problems

Plane Stagnation Circular Cylinder Stagnation



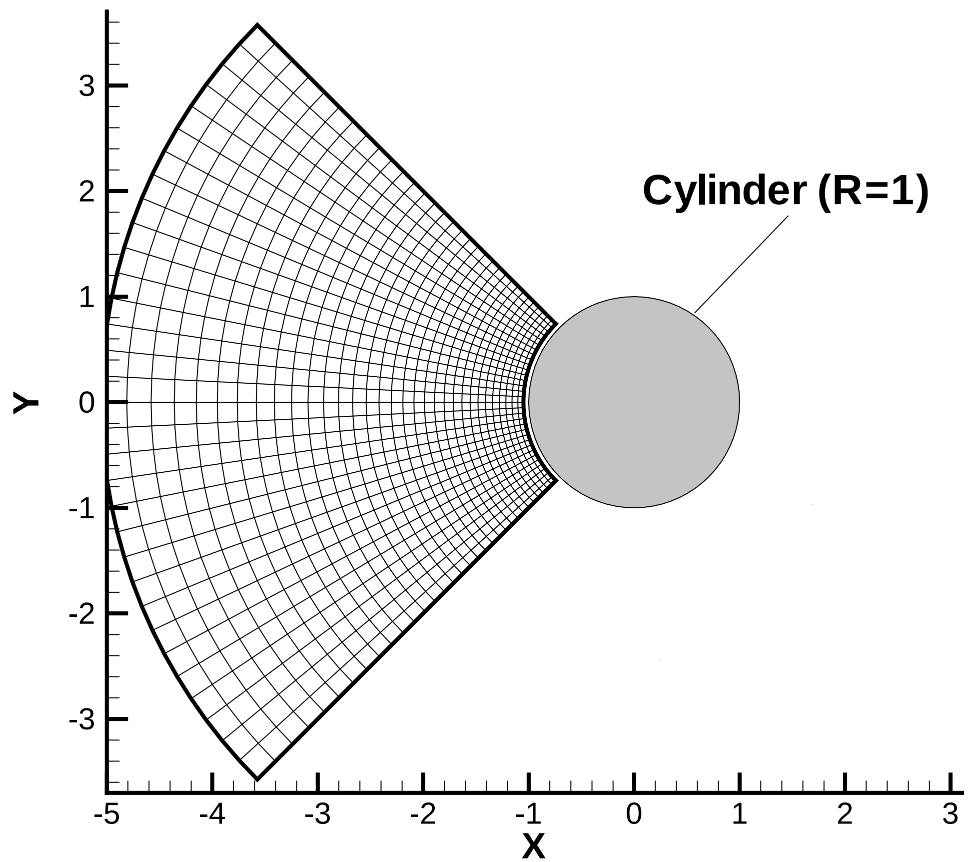
Inviscid Stagnation Flow : Boundary Conditions

- Differential Conditions I - primitive equation set:
  - Inflow : Given u, v
  - Outflow : Given p
  - Tangency : Zero normal velocity
- Differential Conditions II - Pressure-Poisson formulation:
  - Inflow : Continuity equation, $u_x + v_y = 0$
  - Continuity equation enforced only to within discretization error: $Q(u_x + v_y) = 0$
- Other necessary considerations
  - Numerical closure equations
  - Conservation law equations at tangency boundaries

An Example of the Difficulties Near Stagnation

Outflow Boundary Approaching Cylinder

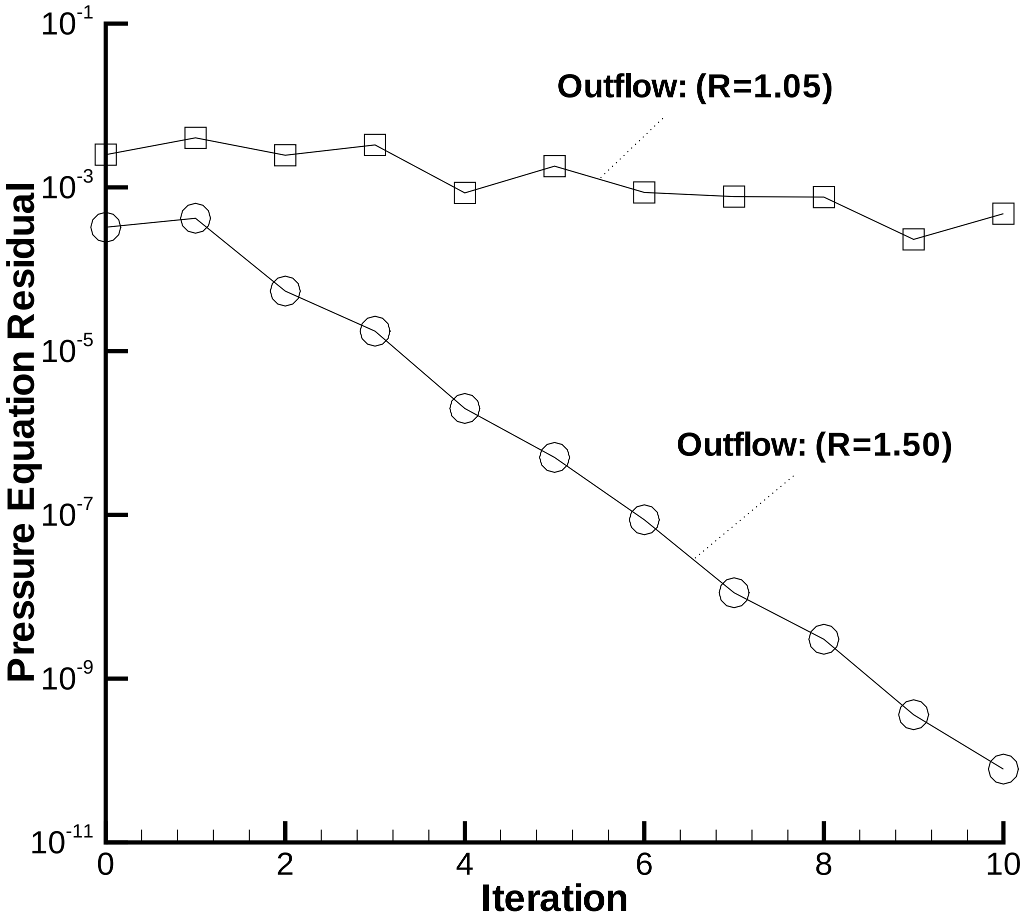
Upper Triangular Solves + Boundary Solves



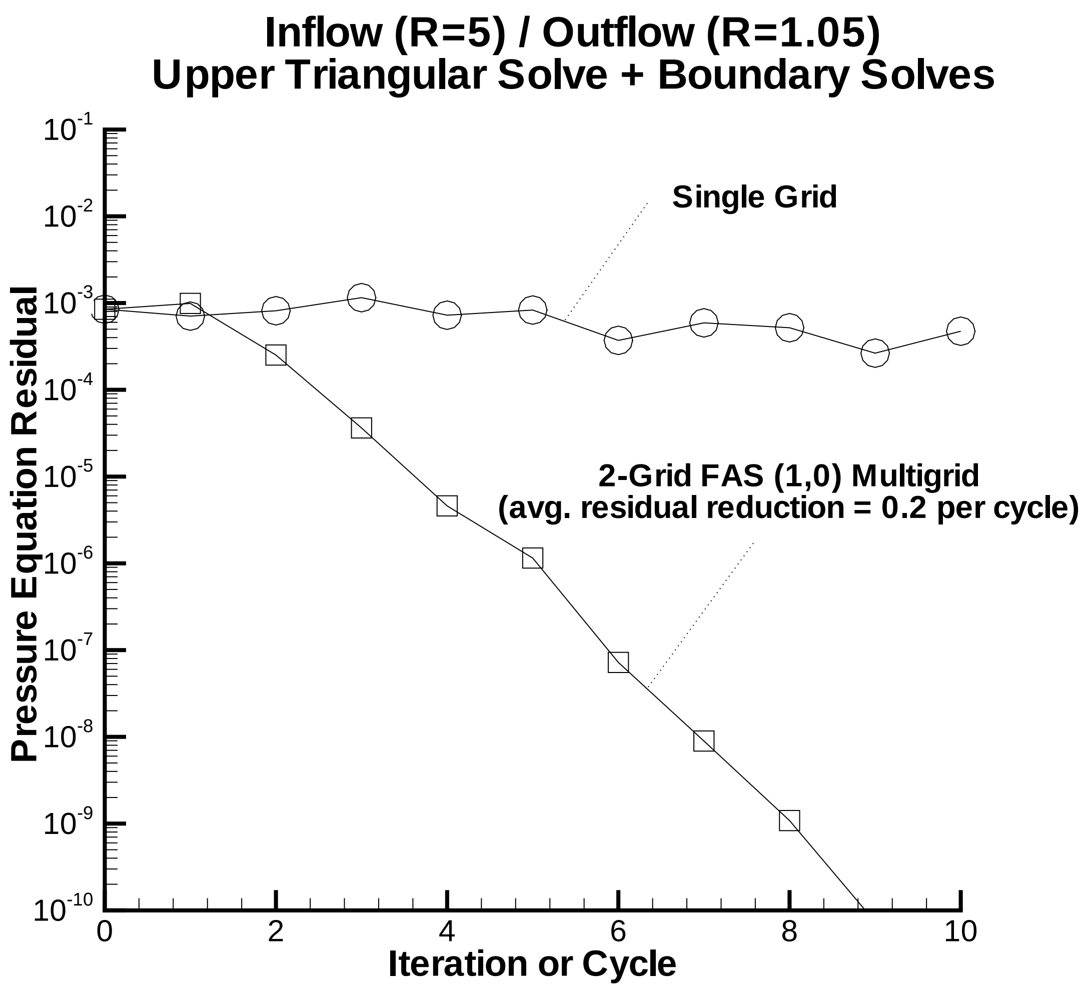
Shown above : Inflow @ R = 5 ; Outflow @ R = 1.05

An Example of the Difficulties Near Stagnation

Outflow Boundary Approaching Cylinder
Upper Triangular Solves + Boundary Solves



An Example of the Difficulties Near Stagnation



Analysis of Relaxation : Plane Stagnation

Exact Solution: $u = -x > 0$, $v = +y > 0$, $u_x = -1$, $v_y = +1$

- Examine behavior of upper triangular **solve** (**L** in **regular flow** + readily available terms)

$$\mathbf{L} = \begin{bmatrix} Q^u + u_x & 0 & \partial_x^c \\ 0 & Q^u + v_y & \partial_y^c \\ 0 & 0 & \Delta^h \end{bmatrix}$$

- Analysis of iteration is a variable coefficient problem

$$\mathbf{L}\,\delta\epsilon \;=\; -\frac{\partial \mathbf{R}}{\partial \mathbf{q}}(\epsilon^n) \qquad \delta\epsilon = \epsilon^{n+1} - \epsilon^n$$

- Several constant coefficient approximations analyzed

Forms of Analysis : Plane Stagnation

- Full space (Fourier) analysis : prediction of possible worst amplifications at initial iterations
- Mode analysis with boundary conditions : prediction of asymptotic convergence rates and penetration distances for characteristic (velocity) direction
  - Differential equations only
  - Periodic in single direction
- Domain size L constrained for relevancy of *constant* coefficient approximation to *variable* coefficient problem

$$|u_x L| \leq u \qquad |v_y L| \leq v$$

Results of Full Space Analysis

- Eigenvalues of error amplification matrix are $f(N, v/u)$:

$$\lambda = \frac{2}{{\theta_x}^2 + {\theta_y}^2}\left[\frac{v_y{\theta_y}^2}{\widehat{Q} + v_y} + \frac{u_x{\theta_x}^2}{\widehat{Q} + u_x}\right] \qquad \theta_x, \theta_y \in [-\pi, \pi]$$

- Uniformly good smoothing rates were found (maximum 1/10)

- In cross-characteristic direction (normal to velocity), the lowest frequency contribution becomes:

$$\lim_{N\to\infty} \widehat{Q} = 0 \quad \Rightarrow \quad \lambda = 2$$

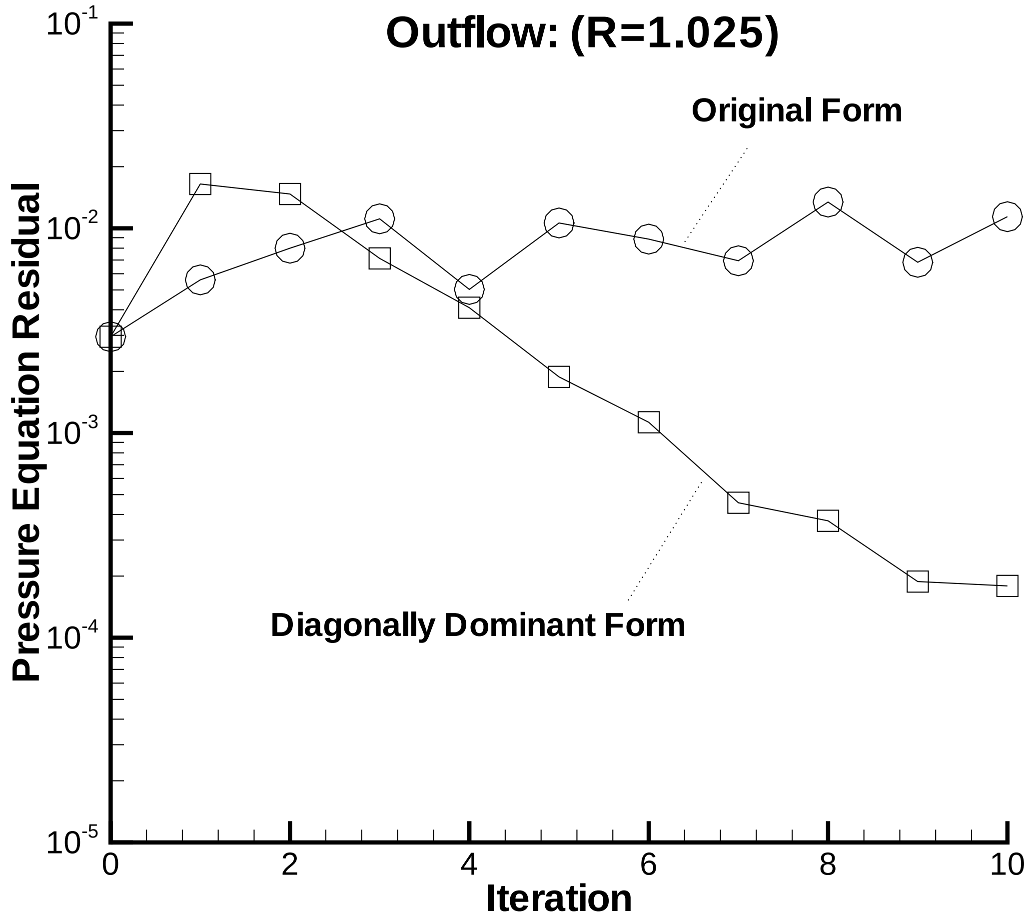
Results of Mode Analysis with Boundary Conditions

- Fast asymptotic error decay for $v = 0$
- Initial error amplified for a few steps
- Analysis prompted assessment of diagonally dominant $\mathbf{L}$ for momentum equations
  - Along symmetry plane, $|\mathbf{L}| = |\frac{\partial \mathbf{R}}{\partial \mathbf{q}}| = (Q - u_x)\Delta$
  - $(\mathbf{L})_{1,1} = Q + |u_x| \qquad (\mathbf{L})_{2,2} = Q + |v_y|$
  - Implementation was somewhat effective in eliminating instability of nonlinear problem (both plane and cylinder stagnation)

An Example of the Difficulties Near Stagnation

Outflow Boundary Approaching Cylinder

Upper Triangular Solves

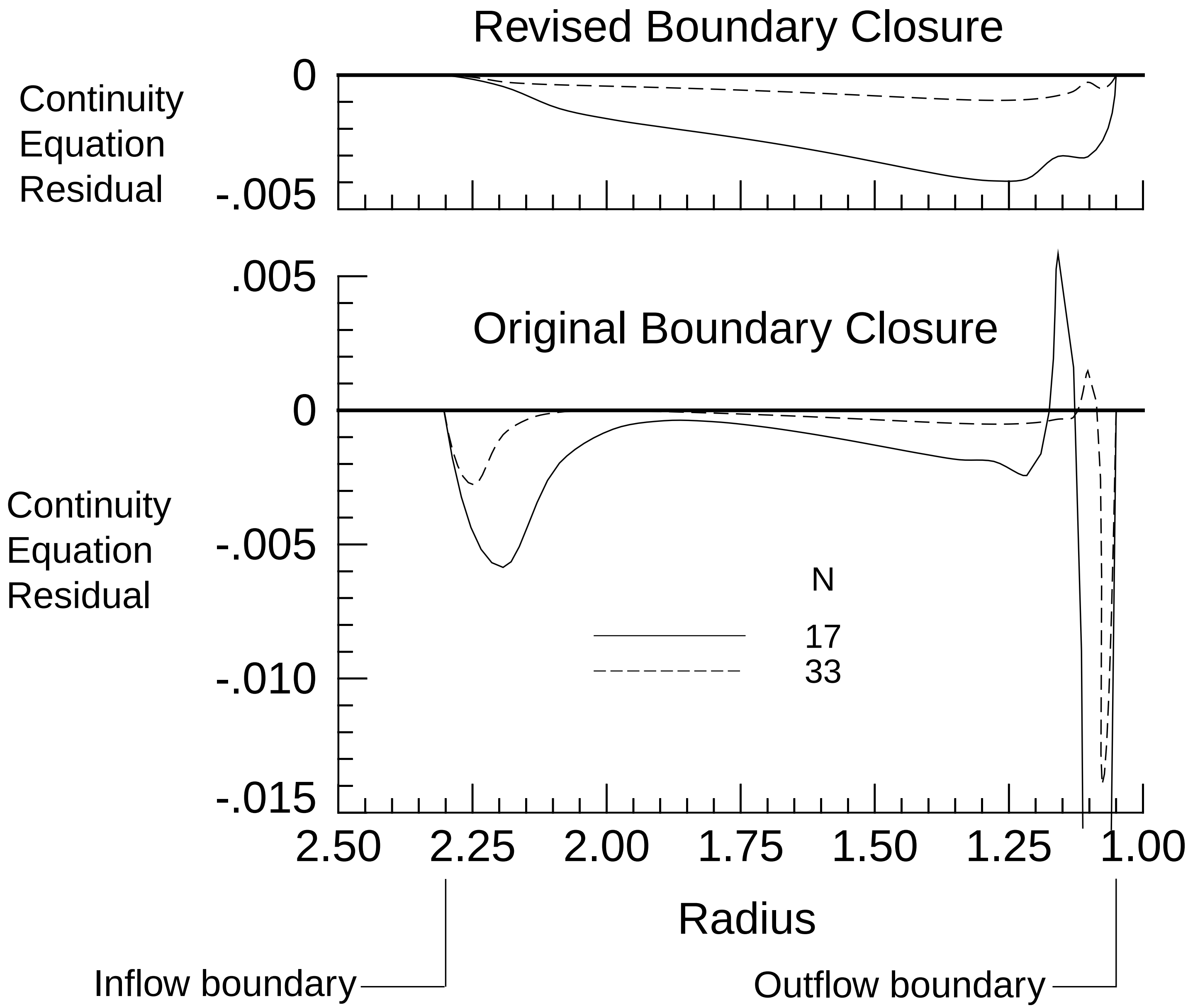


Numerical Closure Equations : Outflow Boundary

(Pressure Specified at $x =$ Constant)

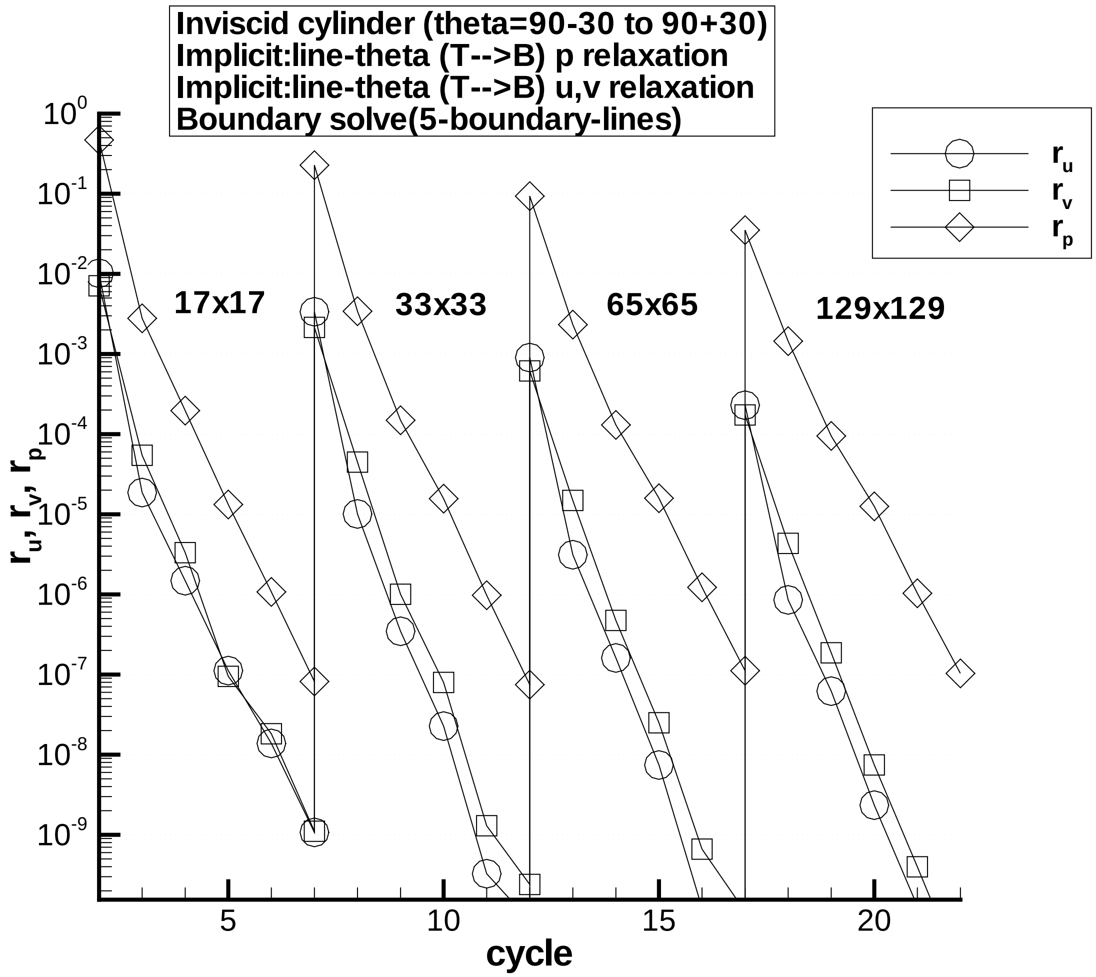
- Original closure :
  - Upwind discretization of momentum equations
    $Q^u u + \partial_x^u p = 0 \qquad Q^v v + \partial_y^c p = 0$
- Revised closure :
  - Upwind discretization of v-momentum equation
    $Q^u v + \partial_y^c p = 0$
  - Compact discretization of continuity equation
    $\partial_x^c u + \partial_y^c v = 0$
- Both conditions yield asymptotically accurate schemes but revised closure allows computation on much coarser grids

Effect of Discrete Boundary Closure



Stagnation Cylinder Flow
Residual Convergence

$FV(2,1)$ cycle



Stagnation Flow
FMG-1 Algorithm, Algebraic Error

Grid	Discretization Error	Alg./Discr. Err
17x17	.320E-03	0.24
33x33	.756E-04	0.36
65x65	.182E-04	0.55
129x129	.456E-05	0.6

Concluding Remarks

- Several issues isolated and improved in stagnation flows
  - Discrete closure conditions at boundary
  - Instability of decoupled solves in deceleration region
- Improved relaxations developed
  - Interior uncoupled line relaxations locally coupled more strongly at boundary
  - Diagonally dominant relaxation for momentum equations
- TME efficiency attained (10 minimum work units)
  - FMG-1 algebraic errors below discretization errors
  - Asymptotic rate of elliptic factor attained