First-Order System Least Squares (FOSLS) for Geometrically-Nonlinear Elasticity

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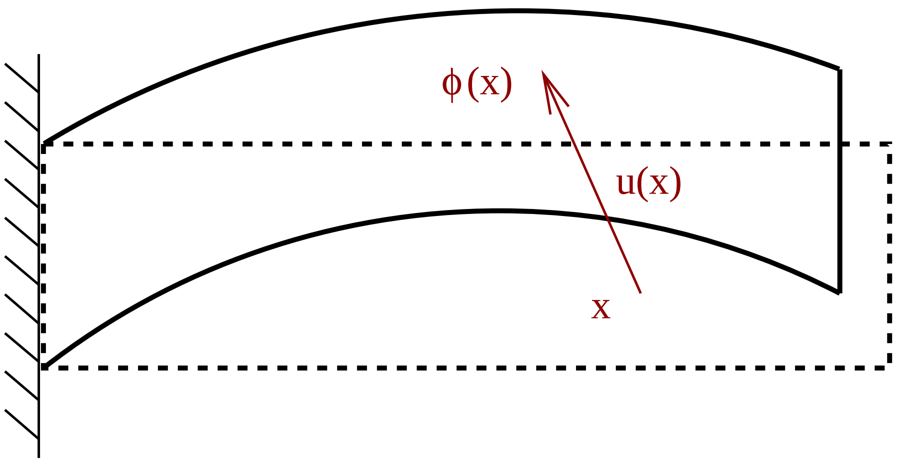
NOTATION AND DEFINITIONS



Displacement, u(x)

Deformation, φ(x)

Strain Tensor, E

Stress Tensor, Σ

$$\phi(x) = x + u(x)$$

$$E = \frac{1}{2}(\nabla u + \nabla u^t + \nabla u^t \nabla u)$$

$$\Sigma = \lambda tr(E)I + 2\mu E$$

NONLINEAR ELASTICITY EQUATIONS

$$\nabla \cdot [(I + \nabla u)\Sigma(\nabla u)] = f \quad \text{in} \quad \Omega$$
$$n \cdot (I + \nabla u)\Sigma(\nabla u) = g \quad \text{on} \quad \Gamma_T$$
$$u = 0 \quad \text{on} \quad \Gamma_D$$

The equilibrium equation may be written as

$$\underbrace{\mu \triangle u + (\lambda + \mu)\nabla\nabla \cdot u}_{linear} + \underbrace{\nabla \cdot P_3(\nabla(u))}_{nonlinear} = f$$

$P_3(\nabla(u))$ is a matrix of degree 3 polynomials of $\nabla(u)$ components

NONLINEAR MODEL

- Use nonlinear strain-displacement equation
- Retain linear material law for generality
- Good for “large-displacement, small-strain” cases
- Very little theory in $W^{k,p}$ spaces for $p = 2$

FOSLS for Linear Elasticity

(Z. Cai, C.O. Lee, T. Manteuffel, S. McCormick, et.al.)

- Linear Model $\Longrightarrow$ "small-displacement, small-strain"
- Introduce $U = \nabla u$, pose as a first order system
- Apply functional minimization principle
- H^1 Elliptic functional
- Results hold for $\lambda \to \infty$ and in 3-D
- Pure displacement, pure traction and mixed boundary conditions

NEWTON-FOSLS

- Pose in terms of $\Phi = \nabla \phi = I + \nabla u$ (First-Order System)
  - Φ is the Jacobian of the mapping
  - $det(\Phi) \approx 1$, $\|\Phi\|_0^2 \approx 2$
- 2-Stage Solution Algorithm
  - Solve for Φ
  - Recover ϕ from: $\nabla \phi = \Phi$
- Linearize around a current approximation by Newton's method
  - Initial guess: $\Phi_0 = I$ ... the reference configuration
  - Newton Step: $\nabla \cdot A(\Phi_n)\Phi = \mathcal{F}$

NEWTON-FOSLS

- Define FOSLS functional for each step:

$$G(\Phi; \Phi_n, \mathcal{F}) = \|\nabla \cdot A(\Phi_n)\Phi - \mathcal{F}\|_0^2 + \|\nabla \times \Phi\|_0^2$$

for Φ in the space

$$\mathbb{V} = \{V \in H^1(\Omega)^4; n \cdot AV = g \;\; on \;\; \Gamma_T, \tau \cdot (V - I) = 0 \;\; on \;\; \Gamma_D\}$$

- Solve by minimization principle:

$$\Phi = \underset{V \in \mathbb{V}}{argmin}\, G(V; \Phi_n, \mathcal{F})$$

ELLIPTICITY OF FUNCTIONAL

Assume sufficient regularity and “small strains” of the solution. Then there exist positive constants c_0 and c_1 such that for all $\Phi \in \mathbb{V}$

$$c_0 \parallel \Phi \parallel_1 \ \leq G(\Phi; 0)^{\frac{1}{2}} \leq \ c_1 \parallel \Phi \parallel_1$$

Can show H^1 Ellipticity for pure displacement boundary conditions

- Existence and Uniqueness in $\mathbb{V}$
- Optimal Multigrid convergence
- Optimal finite element approximation properties

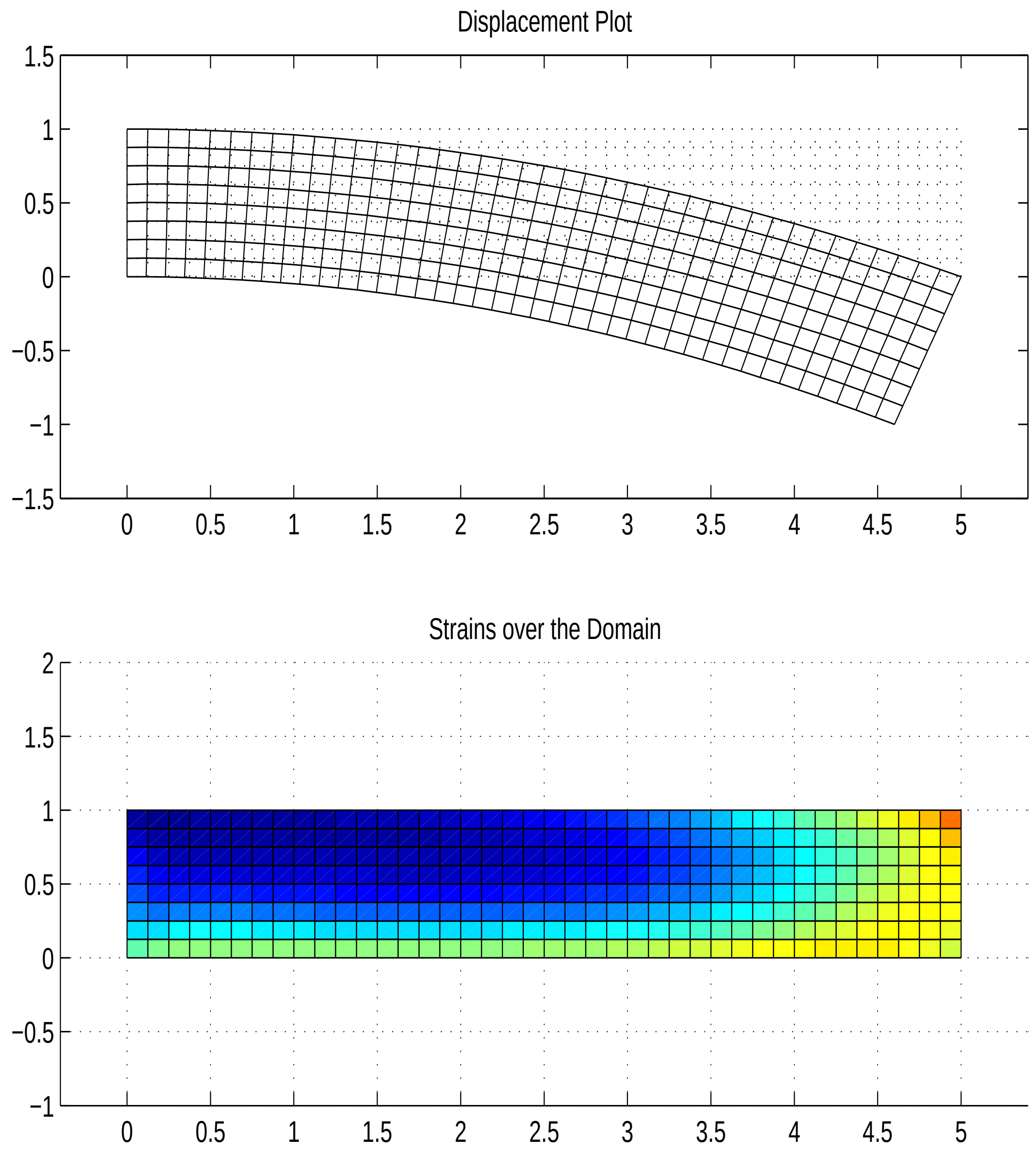
Small strains

$$\max_{\Omega} \left\| \Phi^t \Phi - I \right\|_0 = 0.2263$$

$\tilde{A}$ Bound holds for

$\lambda < 3.25$ or $\nu < 0.38$

- Steel: $\nu = 0.28$
- Nickel: $\nu = 0.30$
- Copper: $\nu = 0.34$
- Aluminum: $\nu = 0.34$



REGULARITY IN NONSMOOTH DOMAINS

Loss of regularity at corners in polygonal domains is well understood (Grisvard et. al.)

$$\boxed{\mathbf{u} = \mathbf{u}_0 + \mathbf{S}(r,\theta)}$$

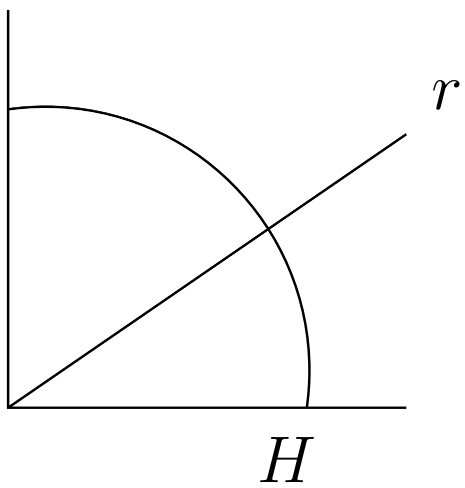
$$\mathbf{u}_0 \in H^2(\Omega)^2$$

$$\mathbf{S}(r,\theta) = r^\alpha \boldsymbol{\Theta}(\theta) \in H^{\alpha+1}(\Omega)^2$$

The value of α is determined by

1. Boundary Condition Type ($\Gamma_D - \Gamma_D, \Gamma_T - \Gamma_T, \Gamma_D - \Gamma_T$)
2. Geometry: interior angle at the corner
3. Lamé Constants: λ, μ

Treatment with Weighted Functionals



$$w_I(r) = \begin{cases} \left(\frac{r}{H}\right)^{-\gamma}, & r \leq H \\ 1, & r > H \end{cases}$$

$$w_O(r) = \begin{cases} \left(\frac{r}{H}\right)^{\frac{\beta}{2}}, & r \leq H \\ 1, & r > H \end{cases}$$

- Weight the unknowns by w_I (choice of γ)
- Weight the equations by w_O (choice of β)

Let $\mathbf{V} = w_I^{-1}\Phi$ (the “smooth” part of Φ)

Weighted Functional: $$\boxed{G_w(\mathbf{V};0) \sim \|w_O \nabla (w_I \mathbf{V})\|_0^2}$$

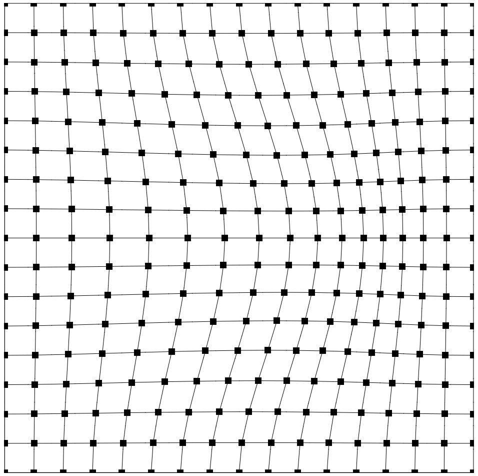
Choose w_I and w_O (i.e., β and γ) to

1. $\mathbf{V} \in H^1(\Omega)$ ... post-process to get $\Phi = w_I \mathbf{V}$
2. FE convergence is $O(h^2)$ with resp. to the functional

NUMERICAL MODEL PROBLEM # 1:

FULL REGULARITY

- Displacement boundary conditions on $\Omega = [0,1]^2$
- Known smooth solution $\Phi \in H^2(\Omega)^4$
- Mesh sizes, $h = \frac{1}{4}, \frac{1}{8}, \ldots, \frac{1}{256}$
- Lamé constants $\lambda = 2.15$, $\mu = 1.0$ (i.e., $\nu = 0.34$)

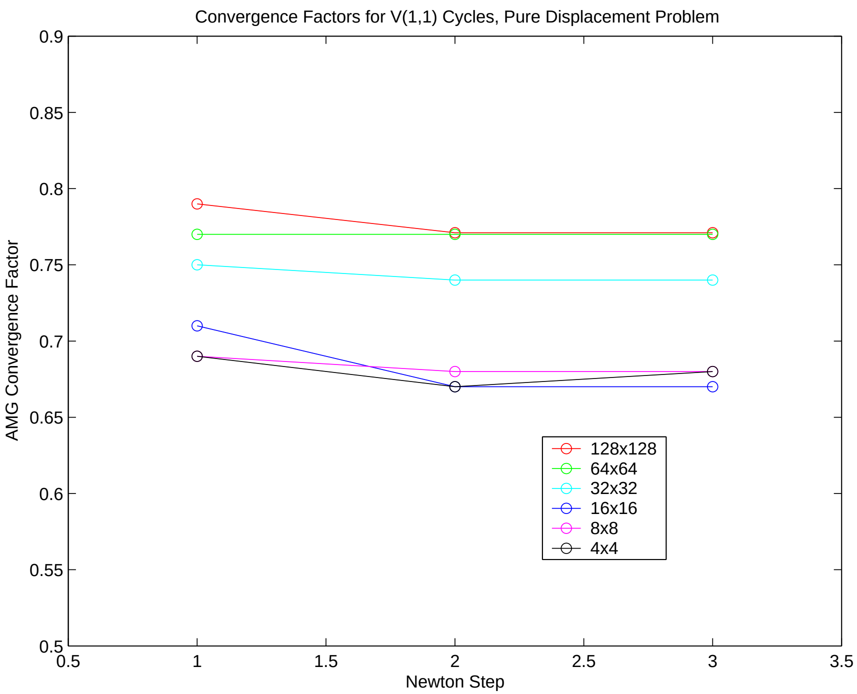


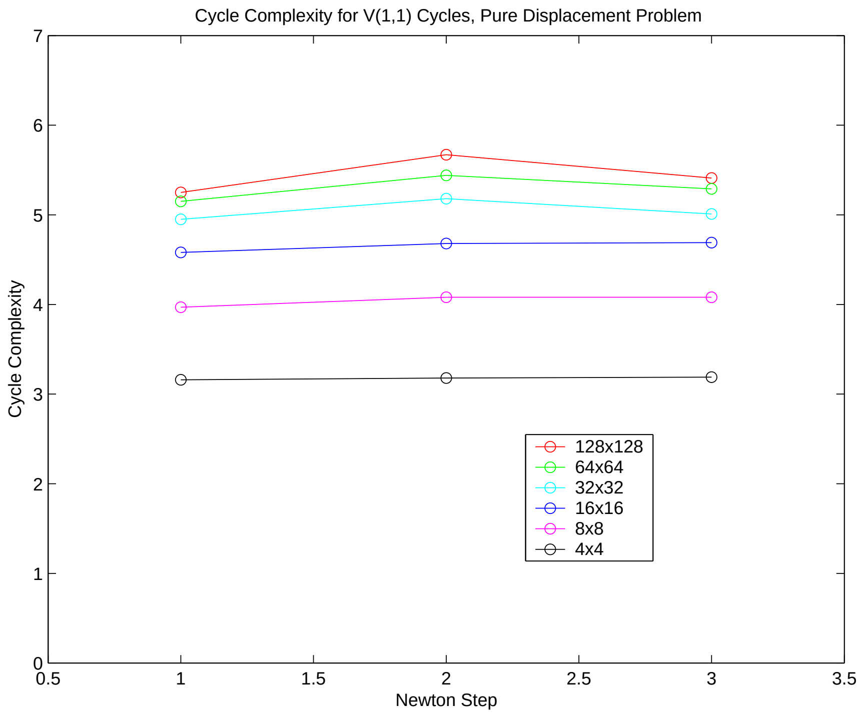
MG convergence factors

Finite Element approximation

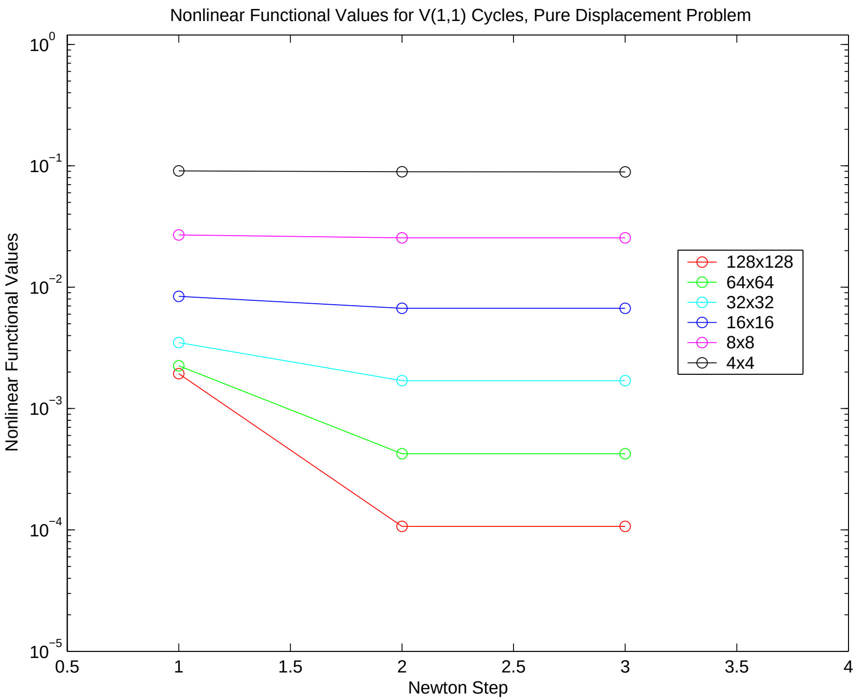
Convergence of Newton's method

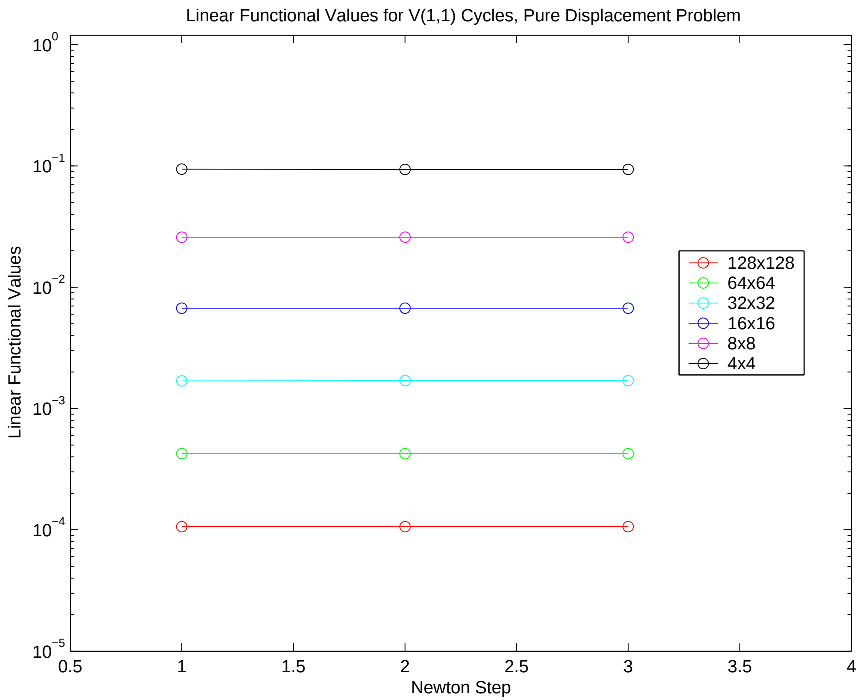
Solver Performance, AMG V(1,1) Cycles





FEM Convergence Properities





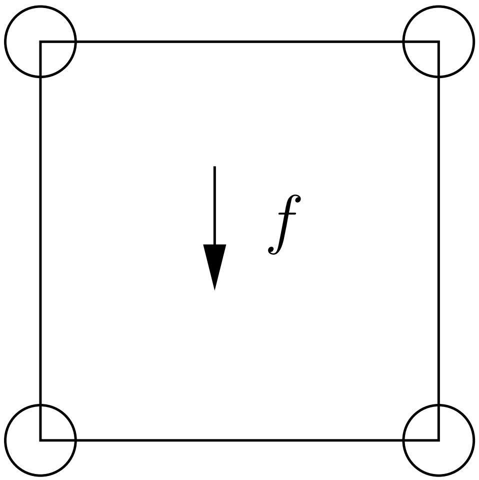
Optimal $O(h^2)$ convergence for $G_{NL}(\Phi; f)$ and $G(\Phi; \mathcal{F})$

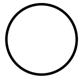
$$G_{NL}(\Phi; f) = \| \nabla \cdot [\Phi \Sigma(\Phi)] - f \|_0^2 + \| \nabla \times \Phi \|_0^2$$

NUMERICAL MODEL PROBLEM # 2:

REDUCED REGULARITY

- Displacement boundary conditions on $\Omega = [0,1]^2$
- Interior body force $\Longrightarrow \Phi \in H^1(\Omega) \backslash H^2(\Omega)$
- Use weighted functional: $G_w(\Phi; \mathcal{F})$
- Recover optimal convergence with $\beta \geq 4 - 2\alpha$





$\Longrightarrow \alpha = 1.37$

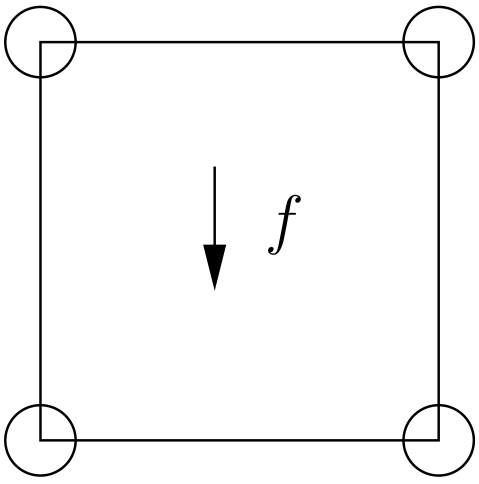
Reduced Regularity: FE Convergence

	No Weighting		Weight: $\beta = 1.26$		$\beta = 2.0$	
Grid	G	Ratio	G_w	Ratio	G_w	Ratio
8	1.39 E-3		1.10 E-3		8.96 E-4	
16	8.37 E-4	1.66	4.41 E-4	2.49	3.31 E-4	2.70
32	4.93 E-4	1.69	1.53 E-4	2.88	1.03 E-4	3.21
64	2.94 E-4	1.68	4.89 E-5	3.12	2.90 E-5	3.55
128	1.76 E-4	1.67	1.50 E-5	3.26	7.81 E-6	3.71
256	1.05 E-4	1.67	4.43 E-6	3.39	2.04e-06	3.83

$O(h^{2(\alpha-1)}) \sim$ 1.67 $\qquad$ $O(h^2) \sim$ 4.00 $\qquad$ $\sim$ 4.00

Numerical Model Problem # 3: Singularities

- Mixed boundary conditions on $\Omega = [0,1]^2$
- Interior body force $\implies \Phi \in L^2(\Omega) \backslash H^1(\Omega)$
- Compute $\mathbf{V} \sim r^\gamma \Phi \in H^1$, $\gamma \geq 1 - \alpha$
- Use weighted functional: $G_w(\mathbf{V}; \mathcal{F})$
- Recover optimal convergence with $\beta \geq 4 - 2\alpha$



$\bigcirc \implies \alpha = 0.69$

SINGULARITIES FE CONVERGENCE

Weight: $\beta = 2.63$ $\beta = 5.0$

Grid	G_w	Ratio	G_w	Ratio
8	1.19 E-3		7.75 E-4	
16	5.65 E-4	2.11	3.18 E-4	2.44
32	2.17 E-4	2.60	1.03 E-4	3.09
64	7.23 E-5	3.00	2.90 E-5	3.55
128	2.24 E-5	3.23	7.65 E-6	3.79
256	6.68 E-6	3.35	1.97 E-6	3.88

$O(h^2) \sim$ 4.00