Multigrid Methods for Stress Concentration problems

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Abstract: Some aspects of the Multigrid Methods implementation to stress concentration problems are discussed. Convenient variational formulations for such problems are proposed. Computational aspects of Richardson's extrapolation methods implementation and nonlinear iterations are investigated. The efficiency of multigrid and some other widespread algorithms is compared.

1 Introduction

The paper concerns some aspects of the stress concentration problems solving. It is known that the perturbations caused by concentrator (micro-hole, crack tip etc.) decay quickly far from it. Therefore, according to the St. Venant principle, we can split the boundary-value problems for the body with the concentrator into two separate problems:

$$(\mathbf{u},\varepsilon,\sigma) = (\mathbf{u},\varepsilon,\sigma)^{I} + (\mathbf{u},\varepsilon,\sigma)^{II}.$$

The first term here means the problem for the "monolithic" body (without concentrators), and the second one represents the concentrator on unbounded domain, where the boundary conditions in the point at infinity are the same as a local stress state in the first problem $(\mathbf{u},\varepsilon,\sigma)^{I}$ at the location of the concentrator. The present work deals with the second type of problems, that is similar to boundary layers (cf. [9], [10]) The main efforts are devoted to the constructing of the suitable variational formulations of the problems and fast multigrid algorithms implementation.

The paper is organized as follows. In section 2 we briefly describe the main linear elasticity boundary-value problem. Section 3 concerns the aspects of weak (variational) formulation of considered problems. Some new suitable formulations for the stress concentration problems are introduced. As model problems we discuss the following two-dimensional ones: plane deformation and anti-plane share of unbounded domain with circular hole; anti-plane stress-strain state near crack tip. In section 4 we consider the main multigrid algorithms: coarse-grid correction cycle and nested iterations. Section 5 describes the multigrid algorithms with Richardson's extrapolation that allow us to obtain the solution of higher-order accuracy on the basis of simple piecewise linear approximations. In section 6 we deal with non-linear multigrid algorithms. We consider here the problem of a simple plasticity theory: small elasto-plastic strains. In section 8 some numerical results are given.

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2 Linear elasticity problems

We will consider the linear elasticity boundary-value problem formulation as follows: for a given domain $\Omega \subset R^d$, d=2,3 find the displacements vector-function $\mathbf{u}$: $\Omega \to R^d$, that satisfies to Lame' equations and boundary conditions:

$$\begin{aligned} &(\lambda+\mu)\nabla div\mathbf{u}+\mu\Delta\mathbf{u}+\rho\mathbf{F}=0, \\ &\mathbf{u}=\mathbf{u}_0 \quad \text{on } \Gamma_0 \subset \Gamma=\partial\Omega, \\ &\sigma_{ij}\nu_j=q_i \quad \text{on } \Gamma_1=\Gamma\backslash\Gamma_0, \end{aligned} \tag{1}$$

Note that the summation over the repeated indexes should be done. Here λ,μ are Lame' parameters; $\mathbf{F}$, $\mathbf{q}$ are vectors of volume and surface load respectively, ν is outer normal to the boundary of Ω; σ and ε are stress and strain tensors that determines as

$$\varepsilon_{ij}(\mathbf{u})=\frac{1}{2}\left(\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i}\right) \text{(Cauchy' equations)},$$

$$\sigma_{ij}=\lambda\theta\delta_{ij}+2\mu\varepsilon_{ij},\ \theta=vol(\varepsilon)=\varepsilon_{ii} \text{ (generalized Hook's law)}.$$

3 Variational formulations

For the finite element method implementation we use the generalized variational formulation: find $\mathbf{u}\in S_A=\{\mathbf{u}\in (W_2^1(\Omega))^d: \mathbf{u}=\mathbf{u}_0 \text{ on } \Gamma_0\}$, $W_2^1(\Omega)$ is Sobolev's space, that the following equation is valid

$$L(\mathbf{u},\mathbf{v})=f(\mathbf{v}), \quad \forall\mathbf{v}\in S_B=\{\mathbf{v}\in (W_2^1(\Omega))^d: \mathbf{v}=0 \text{ on } \Gamma_0\}, \tag{2}$$

where bilinear form L and functional f are determined as follows

$$L(\mathbf{u},\mathbf{v})=\int_\Omega \left(\lambda div\mathbf{u}\, div\mathbf{v}+2\mu\varepsilon_{ij}(\mathbf{u})\varepsilon_{ij}(\mathbf{v})\right) d\Omega, \tag{3}$$

$$f(\mathbf{v})=\int_\Omega \rho\mathbf{F}\mathbf{v}\,d\Omega+\int_{\Gamma_1}\mathbf{q}\mathbf{v}\,d\gamma$$

For the stress concentration problems we will use several special variational formulations. It seems to be useful the following approach to this problem. Let's introduce some coordinate transformation from the original domain Ω to a simple "canonical" domain D (e.g. Unit Square, Cube etc.). More precisely we are interested in the inverse conversion $\aleph: D\to\Omega$. We introduce in D a uniform grid, and the mapping $\aleph$ will produce in Ω the corresponding curvilinear grid. $\aleph$ should provide the proper grid refinement (in Ω) near concentrator in order to take into account the behavior of the solution. Let us consider some typical two-dimensional cases in more detail.

Plane deformation of unbounded domain with circular hole. For convenient implementation of finite element discretization we will restrict the unbounded domain by a circle with the sufficiently large radius *R*. Suppose that the radius of the hole is *r*. Boundary conditions in the point at infinity we will deflect to the restricting circle. In case of a uniform load on the

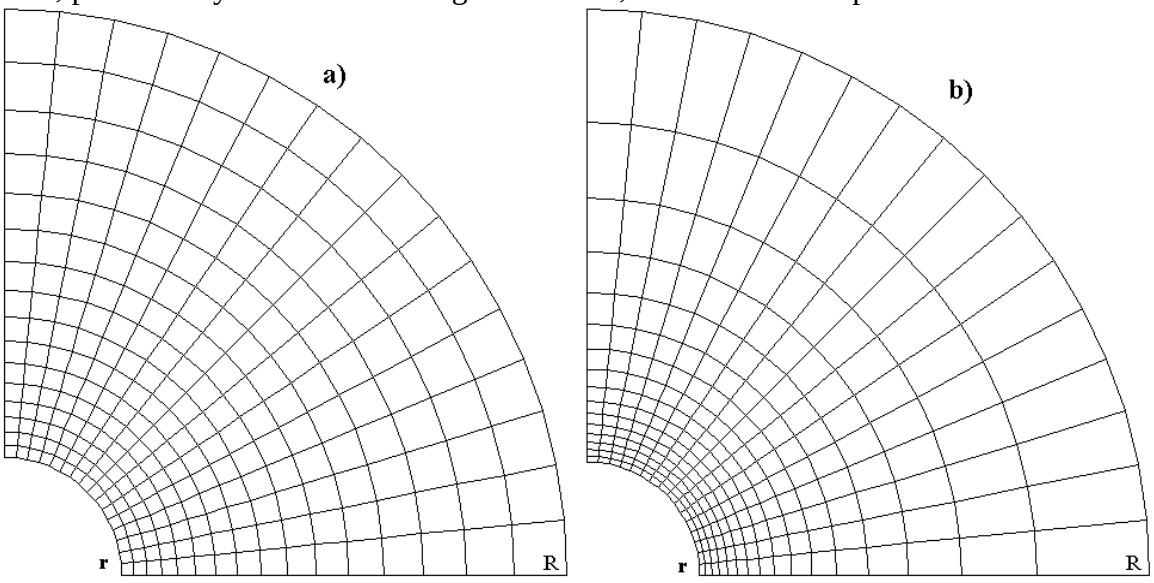
hole surface because of the symmetry it is enough to deal with the domain Ω as a quarter of a ring $\{(\rho,\varphi)$: $r<\rho<R$, $0<\varphi<\pi/2\}$ (in polar coordinates). Then the mentioned coordinate transformation $D\to\Omega$, $D=\{(q_1,q_2)$: $0<q_1<1$, $0<q_2<1\}$, may be presented as

$$\begin{cases} x=\rho(q_1)cos(\alpha q_2) \\ y=\rho(q_1)sin(\alpha q_2) \end{cases}, \quad \alpha=\frac{\pi}{2}, \tag{4}$$

where the coarsening function $\rho(q_1)$ can be chosen by different ways. Note that the stress field perturbations caused by the concentrator are quickly dying out (cf. [6]), therefore we can choose $R\cong 5\div 10r$ without essential loss of accuracy. As long as we are interested in the stress-strain state near hole, we can use the coarsening functions e.g. as follows:

$$\begin{aligned} &\rho(q_1) = r\exp(\alpha\beta q_1),\ \beta = \frac{N_\rho}{N_\varphi} \quad \text{(a)} \\ &\rho(q_1) = \frac{r}{1-\beta q_1},\ \beta = 1-r/R \quad \text{(b)} \end{aligned}, \tag{5}$$

where N_ρ, N_φ determine the grid divisions in ρ and φ directions. Corresponding grids generated in Ω, produced by the uniform triangulation in D, are shown in the picture 1.



Picture 1.

Transformation (4) produces from (2) a new variational problem: find vector-function $\mathbf{u}\in S=\{\mathbf{u}\in(W_2^1(D))^2$: $\mathbf{v}=0$ on $\Gamma_0\}$: $D\to R^2$ satisfying the following equation

$$L_1(\mathbf{u},\mathbf{v})=f_1(\mathbf{v}), \quad \forall\mathbf{v}\in S. \tag{6}$$

Anti-plane shearing of unbounded domain with circular hole. In this deformation only one component of the displacements vector is not equal to zero (for example, $u_1=u_2=0$, $u_3(x_1,x_2)=u(x_1,x_2)\neq 0$). Therefore instead of (3) we obtain the following expression

$$L(u,v)=\int_{\Omega} 2\mu\left\{\frac{\partial u}{\partial x_1}\frac{\partial v}{\partial x_1}+\frac{\partial u}{\partial x_2}\frac{\partial v}{\partial x_2}\right\}dx_1dx_2.$$

Using the same coordinate transformation (4) as in the previous case we obtain from (2) the following formulation: find $u\in S=\{u\in W_2^1(D)$: $u=0$ on $\Gamma_0\}$: $D\to R$ that satisfies the following equation

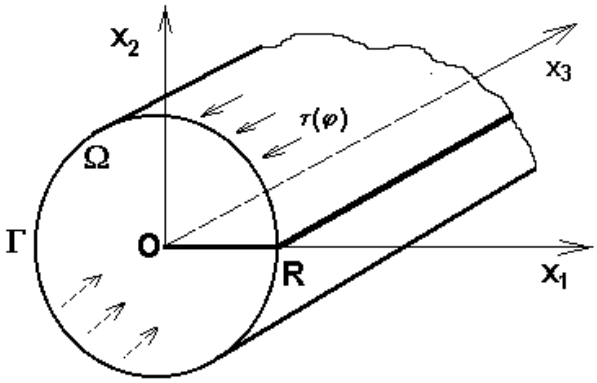
$$L_2(u,v)=f_2(v), \quad \forall v\in S, \tag{7}$$

where the expressions for L_2 and f_2 taking into account the transformation (4), (5a), are as follows

$$L_2(u,v)=\int_{D}\mu\left\{\frac{\alpha}{\beta}(1-\beta q_1)\frac{\partial u}{\partial q_1}\frac{\partial v}{\partial q_1}+\frac{\beta}{\alpha}\frac{1}{1-\beta q_1}\frac{\partial u}{\partial q_2}\frac{\partial v}{\partial q_2}\right\}dq_1dq_2,$$

$f_2(v)=R\alpha\tau\int_0^1 v\cos\alpha q_2\,dq_2$, τ is a shear stress at the infinite point.

<u>Anti-plane stress-strain state near crack tip.</u> Let us consider the infinite cylinder with a radial crack under anti-plane load (picture 2).



Picture 2.

Using coordinate transformation (4), where $\rho(q_1)=Rq_1$, $\alpha=2\pi$, we arrive to the following formulation: $u\in S=\{u\in W_2^1(D)$: $u=0$ on $\Gamma_0\}$: $D\to R$ that satisfies

$$L_3(u,v)=f_3(v), \quad \forall v\in S, \tag{8}$$

$$L_3(u,v)=\int_{D}\mu\left\{\alpha q_1\frac{\partial u}{\partial q_1}\frac{\partial v}{\partial q_1}+\frac{1}{\alpha q_1}\frac{\partial u}{\partial q_2}\frac{\partial v}{\partial q_2}\right\}dq_1dq_2, \quad f_3(v)=R\alpha\int_0^1 v\tau(q_2)dq_2.$$

Note that we can't use such formulation for computations because the form $L_3(u,v)$ infinitely increases near $q_1=0$. Furthermore for the wide class of problems (e.g. crack propagation) it is very important to know the stress intensity factors K_ρ, K_φ, determined by means of first

derivatives of the solution u. The shear stresses $\sigma_{3\rho}$ and $\sigma_{3\varphi}$ have near $\rho=0$ the following behavior:

$$\sigma_{3\rho} = \frac{K_\rho(\rho,\varphi)}{\sqrt{\rho}} + O(1), \quad \sigma_{3\varphi} = \frac{K_\varphi(\rho,\varphi)}{\sqrt{\rho}} + O(1), \quad \rho,\varphi \text{ are polar coordinates.}$$

One of approaches to obtain the satisfactory variational formulation is taking into account the behavior of the solution near the singularity. Namely we will consider the following presentation of the solution:

$$u = \sqrt{q_1}w, \tag{9}$$

where w is a new unknown function. From (8) and (9) we obtain the problem: find such $w\in S=\{w\in W_2^1(D): w=0 \text{ on } \Gamma_0\}: D\to R$ that the following equation is valid

$$L_4(w,v)=f_4(v), \quad \forall v\in S, \tag{10}$$

$$L_4(w,v)=\int_D \mu\left\{\alpha(q_1)^2\frac{\partial w}{\partial q_1}\frac{\partial v}{\partial q_1} + \frac{\alpha q_1}{2}\left(\frac{\partial w}{\partial q_1}v+\frac{\partial v}{\partial q_1}w\right) + \frac{1}{4}\alpha wv + \frac{1}{\alpha}\frac{\partial u}{\partial q_2}\frac{\partial v}{\partial q_2}\right\}dq_1dq_2,$$

$$f_4(v)=R^{3/2}\alpha\int_0^1 v\tau(q_2)dq_2.$$

4 Finite Element formulation

In the domain D we consider the set of grids D_{h_i}, i=0,1,...,p such that each next grid is obtained from the previous one by a suitable refinement procedure, and the corresponding finite-dimensional finite element spaces $S_{h_i}\subset S$. For the problem like (2) we make the corresponding discretized problem: find $\mathbf{u}^{h_i}\in S_{h_i}$ such that

$$L(\mathbf{u}^{h_i},\mathbf{v})=f(\mathbf{v}), \quad \forall \mathbf{v}\in S_{h_i} \tag{11}$$

is valid. These problems are equivalent to the systems of linear equations $\mathbf{L}^{h_i}\mathbf{V}^{h_i} = \mathbf{F}^{h_i}$, where $\mathbf{V}^{h_i}$ are vectors of unknowns, $\mathbf{L}^{h_i}$ are nodal ensemble matrices, $\mathbf{F}^{h_i}$ are right-hand sides. In this paper we use the uniform grids in D and piecewise linear finite elements.

5 Multigrid Methods

Let us consider the standard coarse-grid correction algorithm. For given approximation $\mathbf{u}_0^{h_i}$ (corresponding vector of unknowns is $\mathbf{V}_0^{h_i}$) to the solution $\mathbf{u}^{h_i}$ we obtain the next approach $\mathbf{u}_1^{hi}$ by the following algorithm:

If i=0 directly solve the system $\mathbf{L}^{h_0}\mathbf{V}^{h_0} = \mathbf{F}^{h_0}$ by Gauss method, **else**

$\mathbf{A}_1$. **Smoothing iterations**: $\mathbf{U}_{l+1}= \mathfrak{R}^{h_i}\mathbf{U}_l$, l=0,1,...,m-1, $\mathbf{U}_0=\mathbf{V}_0^{h_i}$, calculation the residuals: $\mathbf{G}^{h_i} = \mathbf{L}^{h_i}\mathbf{U}_m-\mathbf{F}^{h_i}$.

$\mathbf{A}_2$. **Projection of $\mathbf{G}^{h_i}$ into the lower dimensional space**: $\mathbf{G}^{h_{i-1}}=\mathbf{I}_{i-1}^{i}\mathbf{G}^{h_i}$.

$\mathbf{A}_3$. **Coarse grid correction**: obtaining $\mathbf{W}^{h_{i-1}}$ from the system $\mathbf{L}^{h_{i-1}}\mathbf{W}^{h_{i-1}}=\mathbf{G}^{h_{i-1}}$.

$\mathbf{A}_4$. **Interpolation to the finer grid**: $\mathbf{W}^{h_i}=\mathbf{I}_i^{i-1}\mathbf{W}^{h_{i-1}}$.

$\mathbf{A}_5$. **Calculation a new approach**: $\mathbf{V}_1^{h_i}=\mathbf{U}_m-\mathbf{W}^{h_i}$.

$\mathbf{A}_6$. **Post-Smoothing**: $\mathbf{U}_{l+1}= \mathfrak{R}^{h_i}\mathbf{U}_l$, l=0,1,...,m-1, $\mathbf{U}_0=\mathbf{V}_1^{h_i}$, $\mathbf{V}_1^{h_i}=\mathbf{U}_m$.

end

Note that on the step $\mathbf{A}_3$ we can perform one (V-cycle) or two (W-cycle) iterations of the same algorithm **A** on the level i-1. Thus the coarse-grid correction algorithm is formulated recursively. For the full multigrid algorithm we use nested iterations techniques as follows

Obtain $\mathbf{V}^{h_0}$ from the system $\mathbf{L}^{h_0}\mathbf{V}^{h_0} = \mathbf{F}^{h_0}$, using Gauss Method. Let $\mathbf{V}_1^{h_0}=\mathbf{V}^{h_0}$.

For i=1,...,p **do**

$\mathbf{B}_1$. **Interpolation from the coarser grid**: $\mathbf{V}_0^{h_i}=\mathbf{I}_i^{i-1}\mathbf{V}_1^{h_{i-1}}$.

$\mathbf{B}_2$. **Improvement the initial approach using algorithm A**: $\mathbf{V}_0^{h_i}\rightarrow\mathbf{V}_1^{h_i}$.

End

Note, that the convergence (and its speed) of the algorithm B depends on the properties of the form *L* (ellipticity, continuity), proper choice of the relaxation scheme operators $\mathfrak{R}^{h_i}$ and interpolation and projection mappings $\mathbf{I}_i^{i-1}$, $\mathbf{I}_{i-1}^{i}$ (cf. e.g. [3]).

6 Richardson's Extrapolation technique

It seems to be useful the way of obtaining approximate solution of higher-order accuracy based on the simple lower-order schemes (piecewise linear finite element approximation). Such possibilities are offered by extrapolation-like methods (cf. [5]). These methods consist of the approximate solutions calculation on the grids $D_{h_{i-2}}$ and $D_{h_{i-1}}$ and then building the higher-order solution on the grid D_{h_i} as a proper linear combination of them. Multigrid methods with Richardson's extrapolation can be developed, preserving the advantages of the both approaches: higher-order accuracy and $O(N_i)$ computational work, where N_i is number of unknowns in the problem (11) on the level i.

For i≤1 perform **B** algorithm.
For i=2, ..., p **do**

C$_1$. **Cubic interpolation from the two coarser grids and calculation the linear combination**:

$$\mathbf{V}_0^{h_i} = \frac{5}{4}\tilde{I}_i^{i-1}\mathbf{V}_1^{h_{i-1}} - \frac{1}{4}\tilde{I}_i^{i-2}\mathbf{V}_1^{h_{i-2}}.$$

C$_2$. **Improvement the initial approach using algorithm A**: $\mathbf{V}_0^{h_i} \rightarrow \mathbf{V}_1^{h_i}$.

End

On the finest grid we calculate the approximate solution as a linear combination $\mathbf{V}_1^{h_p} = \frac{3}{4}\mathbf{V}_1^{h_p} - \frac{1}{3}\tilde{I}_p^{p-1}\mathbf{V}_1^{h_{p-1}}$.

Here $\tilde{I}_i^{i-1}$ and $\tilde{I}_i^{i-2}$ are higher-order prolongation operators. Note that the convergence proof of the Richardson's extrapolation schemes is based on the following expansion:

$$\mathbf{u}^{h_i}(x) = \mathbf{u}^I(x) + h_i^2\mathbf{v}^I(x) + h_i^4\xi^{h_i}(x), \quad \forall x \in D,$$

where $\mathbf{u}^I(x)$ is a piecewise linear prolongation of the exact solution **u** of the differential problem to S_{h_i}, grid function $\mathbf{v}^I(x)$ is the interpolant on D_{h_i} of some smooth function **v**, that does not depend on i. The convergence order of extrapolation algorithm depends on the remainder term ξ^{h_i} estimation. In case of $\|\xi^{h_i}\|_{L_2(D)} \le c_1$ it can be proved (cf. [5],[7]) the following appreciation:

$$\|\mathbf{u} - \tilde{\mathbf{v}}^{h_p}\|_{L_2(D)} \le c_2 h_p^4, \qquad (12)$$

where $\tilde{\mathbf{v}}^{h_p}(x)$ is a piecewise cubic prolongation of the approximate solution $\mathbf{u}_1^{h_p}(x)$ obtained as a result of algorithm C. Naturally in case of poor estimation for ξ^{h_i} we obtain the worse appreciation than (12) (value of c_2 won't be a constant).

7 Nonlinear methods

Let us consider the deformation plasticity theory boundary-value problem. Instead of the first equation (1) we will have

$$\frac{\partial}{\partial x_i}\left\{\tilde{\lambda}(\mathbf{u})\mathrm{div}\mathbf{u}\right\} + \frac{\partial}{\partial x_j}\left\{\tilde{\mu}(\mathbf{u})\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)\right\} = F_i, \quad i=1,...,d,$$

where

$\tilde{\mu}(\mathbf{u})=\Phi(\varepsilon_i(\mathbf{u}))$, $\tilde{\lambda}(\mathbf{u})=K-\frac{2}{3}\Phi(\varepsilon_i(\mathbf{u}))$,

$\varepsilon_i(\mathbf{u})=\sqrt{\frac{2}{3}\left[\varepsilon_{ij}(\mathbf{u})\varepsilon_{ij}(\mathbf{u})-\frac{1}{3}(div\mathbf{u})^2\right]}$, (Strain intensity)

$\Phi(\varepsilon_i(\mathbf{u}))=\sigma_i(\varepsilon_i)/3\varepsilon_i$,

$\sigma_i=\sigma_i(\varepsilon_i)$, (Hardening function)

$K=\lambda+\frac{2}{3}\mu$. (Volume elasticity module)

The corresponding variational problem can be written as follows

$$\hat{L}(\mathbf{u},\mathbf{v}) = f(\mathbf{v}), \qquad \forall \mathbf{v}\in \mathbf{S}_B,$$

where the form $\hat{L}$ is linear with only second argument

$$\hat{L}(\mathbf{u},\mathbf{v})=\int_\Omega\left\{\tilde{\lambda}(\mathbf{u})div\mathbf{u}\ div\mathbf{v}+2\tilde{\mu}(\mathbf{u})\varepsilon_{ij}(\mathbf{u})\varepsilon_{ij}(\mathbf{v})\right\}d\Omega.$$

For efficient multigrid algorithms implementation we should use some linearization method. In this paper we consider two popular methods (cf. [4],[1]):

Fictitious loads iterations:

$$L(\mathbf{u}^{k+1},\mathbf{v})=L(\mathbf{u}^k,\mathbf{v}) - \omega(\hat{L}(\mathbf{u}^k,\mathbf{v})-f(\mathbf{v})), \qquad (13)$$

Variable elasticity parameters iterations:

$$L(\mathbf{u}^{k+1},\mathbf{v};\mathbf{u}^k)=L(\mathbf{u}^k,\mathbf{v};\mathbf{u}^k) - \omega(\hat{L}(\mathbf{u}^k,\mathbf{v})-f(\mathbf{v})),\ k=0,1,\ldots . \qquad (14)$$

Here $\omega>0$ is an iteration coefficient; bilinear form $L(\mathbf{u},\mathbf{v};\mathbf{w})$ depends on the function $\mathbf{w}$ as a parameter:

$$L(\mathbf{u},\mathbf{v};\mathbf{w})=\int_\Omega\left\{\tilde{\lambda}(\mathbf{w})div\mathbf{u}div\mathbf{v}+2\tilde{\mu}(\mathbf{w})\ \varepsilon_{ij}(\mathbf{u})\varepsilon_{ij}(\mathbf{v})\right\}d\Omega.$$

Instead of (11) we will have the following finite element formulation

$$\hat{L}(\mathbf{u}^{h_i},\mathbf{v})=f(\mathbf{v}), \quad \forall \mathbf{v}\in S_{h_i} \qquad (15)$$

and corresponding to (13), (14) linearized grid problems. For successful multigrid methods constructing it is important the convergence of the linearization processes like a geometric progression

$$\|\mathbf{u}^{h_i}-\mathbf{u}^{h_i}_{k+1}\|_{\mathbf{S}_A^{h_i}} \le q\ \|\mathbf{u}^{h_i}-\mathbf{u}^{h_i}_{k}\|_{\mathbf{S}_A^{h_i}}, \qquad (16)$$

where the multiplier q<1 does not depend on k. For some restrictions on the hardening function $\sigma_i=\sigma_i(\varepsilon_i)$ such convergence for the process (13) can be proved (cf. [11]). Note that for the method (14) better estimation can be obtained

$$q = c_3 h_i^{\alpha/2},\ \alpha>0,$$

in the norm $\|\mathbf{v}\|_L = (L(\mathbf{v},\mathbf{v};\mathbf{u}^{h_i}))^{1/2}$, but for initial approach belonging to some sphere

$$\mathbf{B} = \left\{\mathbf{v}\in \mathbf{S}_A^{h_i}: \|\mathbf{v}-\mathbf{u}^{h_i}\|_L \le c_4 h_i^{\alpha}\right\},$$

cf. [7], that is proved to be true in practice.

And now let us formulate the nonlinear version of the multigrid algorithm obtaining an approximate solution $\tilde{\mathbf{u}}^{h_i}$ *on the level of the truncation error*

$$\|\tilde{\mathbf{u}}^{h_i}-\mathbf{u}^{h_i}\|_{\mathbf{S}_A^{h_i}} \le c_5 h_i^2,\quad i=0,1,...,p. \qquad (17)$$

Because of the small number of unknowns we can achieve the estimation (17) on the coarsest grid (i=0) spending less computational work.
For k=1,...,p **do**

D$_1$. **Interpolation from the coarser grid**: $\tilde{\mathbf{u}}^{h_i}=\mathbf{I}_i^{i-1}\tilde{\mathbf{u}}^{h_{i-1}}$.

D$_2$. **Perform n iterations of the process (13)**. Use algorithm **A** (W-cycle) to improve the approximate solution $\mathbf{u}^{h_i}$ on the level i.

end

THEOREM (cf. [8]). Suppose the equation (16) and the conditions of the algorithm **A** convergence is valid. Then the number of nonlinear iteration **n** in the algorithm **D** exists and does not depend on i, that the equation (17) is valid. The total amount of the computational work in the algorithm **D** is $O(N_p)$, where N_p is number of unknowns on the finest grid.

8 Numerical results

Because of difficulties in obtaining precise analytical estimations we use some approximate techniques like local mode analysis, a posteriori error estimation etc. Local mode (Fourier) method tries to assess the norm of the amplitude reduction matrix $\mu(\theta^1,\theta^2)$:

$$\Re^{h_i}\Theta^i(\theta^1,\theta^2) = \mu(\theta^1,\theta^2)\Theta^i(\theta^1,\theta^2),$$

where $\Re^{h_i}$ is a relaxation scheme operator, $\Theta^i(\theta^1,\theta^2)$ - Fourier components (harmonics). For the relaxation scheme quality estimation the *smoothing rate* μ_0 is introduced (cf. [2])

$$\mu_0 = \max_{\pi/2\le|\theta^1|,|\theta^2|\le\pi} \|\mu(\theta^1,\theta^2)\|.$$

Such parameter thus depends on the grid point. To assess the smoothing property in the entire domain we consider the *total smoothing rate*

$$\bar{\mu}_0 = \max_{x \in \Omega_{h_i}} \mu_0(x).$$

The less the value of $\bar{\mu}_0$ the better the quality of the operator $\Re^{h_i}$ as a smoother. We apply the local mode analysis to the stress concentration problems considered above to choose the suitable relaxation scheme. For the problem (5), the following results are obtained.

R/r	Relaxation scheme					
	A	B	C	D	E	F
2	1.833	0.927	0.456	0.460	0.928	0.927
3	1.894	0.877	0.667	0.655	0.877	0.875
5	1.769	0.880	0.878	0.879	0.831	0.827
10	1.812	0.958	0.956	0.958	0.795	0.790
20	1.830	0.962	0.963	0.962	0.776	0.769
30	1.835	0.977	0.978	0.976	0.770	0.762

Here A is a designation for the Jacobi relaxation, B denotes the pointwise Gauss-Seidel relaxation. C and D mean two variants of blockwise y-line relaxation. In case of C we simultaneously change the both components of the displacements vector in each node of the y-line block. In the case D we consecutively build the y-line blocks consisting of only one component of the displacements vector. E and F are the corresponding variants of the x-line relaxation. These results show that the relaxations A-D are degenerate when *R* increases, but the x-line relaxations preserve more or less satisfactory smoothing capabilities.

Let us consider the question about the suitable number of smoothing iterations in the coarse-grid correction algorithm. The results are given in the following table

Number of iterations	Grid $N_1 \times N_2$	Error (L_2-)norms		
		$\|\mathbf{u}-\tilde{\mathbf{u}}^{h_i}\|$	$\|\mathbf{u}^{h_i}-\tilde{\mathbf{u}}^{h_i}\|$	$\|\mathbf{u}-\mathbf{u}^{h_i}\|$
m=2	7×7	8.80×10^{-2}	5.33×10^{-2}	4.33×10^{-2}
	13×13	2.61×10^{-2}	2.37×10^{-2}	9.74×10^{-3}
	25×25	2.99×10^{-3}	2.51×10^{-3}	2.68×10^{-3}
	49×49	6.79×10^{-4}	4.27×10^{-4}	6.95×10^{-4}
m=4	7×7	7.65×10^{-2}	4.07×10^{-2}	4.33×10^{-2}
	13×13	1.88×10^{-2}	1.56×10^{-2}	9.74×10^{-3}
	25×25	2.52×10^{-3}	9.69×10^{-4}	2.68×10^{-3}
	49×49	6.90×10^{-4}	9.62×10^{-5}	6.95×10^{-4}
m=6	7×7	7.21×10^{-2}	3.59×10^{-2}	4.33×10^{-2}
	13×13	1.66×10^{-2}	1.29×10^{-2}	9.74×10^{-3}
	25×25	2.52×10^{-3}	7.00×10^{-4}	2.68×10^{-3}
	49×49	7.09×10^{-4}	3.92×10^{-5}	6.95×10^{-4}

We compare here precise analytical solution $\mathbf{u}$ of the differential model problem (6), precise solution $\mathbf{u}^{h_i}$ of the discrete problem and the approximate multigrid solution $\tilde{\mathbf{u}}^{h_i}$. We can see that it is enough m=2 smoothing iterations to achieve the level of truncation error. The results of applying the Richardson's extrapolation algorithms are shown below.

Methods	Error norm $\|\mathbf{u}-\tilde{\mathbf{u}}^{h_i}\|$				
	4×4	7×7	13×13	25×25	49×49
Without extrapolation	1.82×10^{-1}	8.80×10^{-2}	2.61×10^{-2}	2.99×10^{-3}	6.79×10^{-4}

With extrapolation	1.22×10^{-1}	2.29×10^{-2}	1.02×10^{-2}	1.93×10^{-3}	1.32×10^{-4}

The results indicate that the extrapolation yields the accuracy improvement, but worse than the estimation (12).

And now we consider the comparison between the efficiency of the multigrid method and several popular widespread algorithms ("control" methods).

Methods	Computational work			Time, sec		
	13×13	25×25	49×49	13×13	25×25	49×49
1	10	17	24.5	0	1	4
2	101.5	262	568.4	1	20	250
3	192	738	2354	4	50	1034
4	-	-	-	1	9	122

Multigrid method is denoted here as 1. Method 2 is an improved variant of upper relaxation algorithm where the initial approximation is taken from the coarser grid and the relaxations on the finer grid perform until the level of truncation error is reached. Simple upper relaxation method on the finest grid is denoted as 3. Method 4 is direct Gauss method.

Similar results occur in case of the problems (7), (10). We put out here only the results concerning the Richardson's extrapolation implementation for the problem (10).

Methods	Error norm $\|\mathbf{u}-\tilde{\mathbf{u}}^{h_i}\|$				
	4×4	7×7	13×13	25×25	49×49
Without extrapolation	1.19×10^{-2}	3.71×10^{-3}	1.41×10^{-3}	5.19×10^{-4}	1.88×10^{-4}
With extrapolation	1.19×10^{-2}	2.87×10^{-3}	9.71×10^{-4}	3.56×10^{-4}	1.26×10^{-4}

These results show us that the extrapolation doesn't improve the accuracy. It needs to change the variational formulation in order to obtain better estimation for the remainder term ξ^{h_i}.

Let us discuss the nonlinear multigrid methods implementation now. Note that in this case we can not precisely calculate the error norms because we don't know the analytical solution. Therefore we should use some indirect error estimators. In this paper we consider the simple way dealing with the residuals norm assessment. In fact for this reason we should come up with an estimate for the minimal eigenvalue of the differential (or discrete) problem. Such estimate can be (approximately) obtained from the corresponding linear elasticity problem.

The results of comparison the computational efficiency for nonlinear problem like (6), (7) are given in the following tables.

Tensile load

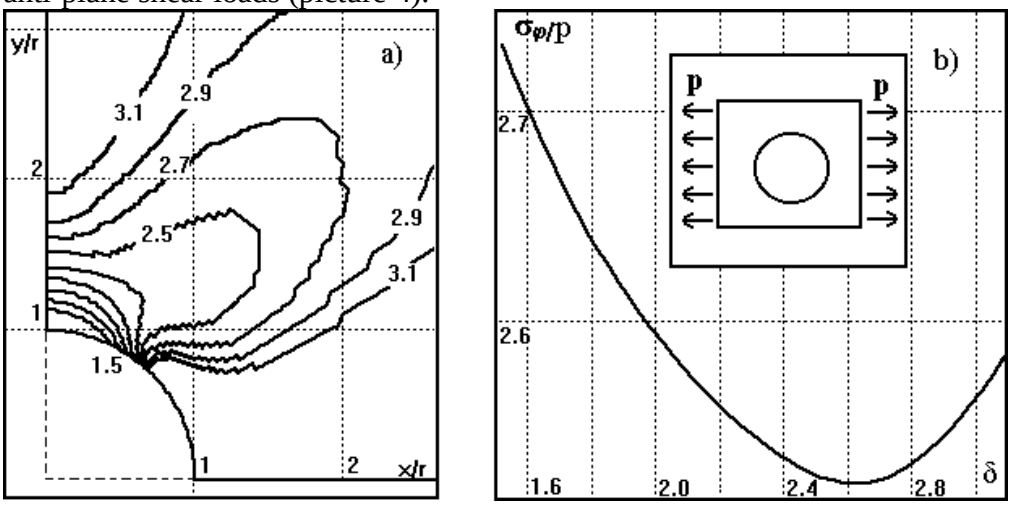
Methods	Computational work			Time, sec		
	13×13	25×25	49×49	13×13	25×25	49×49
1	45	76	110.25	1	6	34
2	347	308	248.25	12	73	239
3	224	798	2330	30	432	6613

Anti-plane shear load

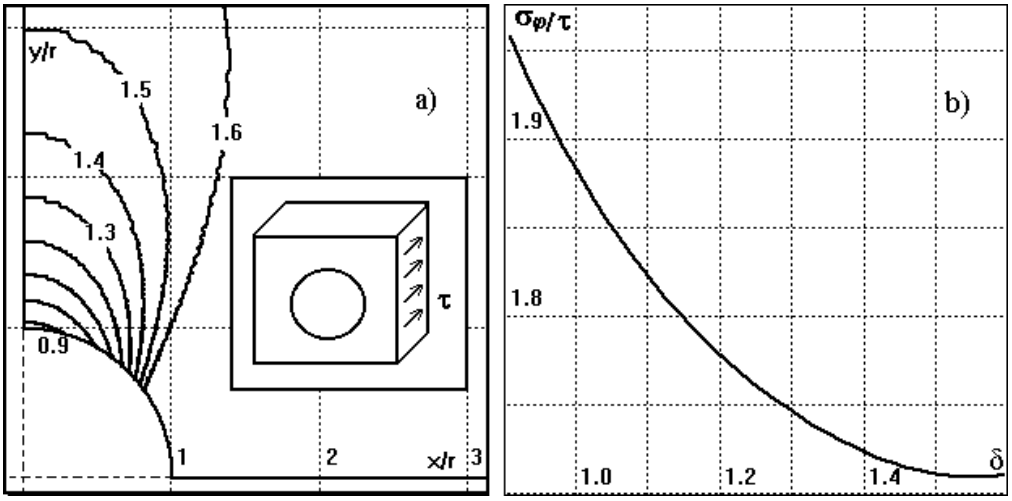
Methods	Computational work			Time, sec		
	13×13	25×25	49×49	13×13	25×25	49×49
1	46	71	95.5	1	6	33
2	308.75	285.38	452.69	7	61	1113
3	432	1235	5156	70	823	19588

1 . *Multigrid method*
2. *Improved upper relaxation (initial approach takes from the coarser grid)*
3. *Simple upper relaxation (on the finest grid)*

Below we put out some numerical results for elasto-plastic stress concentration problems. Diagrams below show the elasto-plastic strain areas development (a) and the stress concentration rate variation (b) depending on dimensionless loading parameter δ for the tensile (picture 3) and anti-plane shear loads (picture 4).



Picture 3



Picture 4

References

[1] I.A. Birger. *Some general methods of plasticity theory problems solving*: Applied Mathematics and Mechanics, vol. 15, 1951, 6, pp. 765-770 (Russian)

[2] A. Brandt. *Guide to Multigrid Development*: Lecture Notes in Math., vol. 960, 1982, pp 220-312

[3] W. Hackbusch. *Multigrid Methods and Applications*: Berlin, N.Y., Springer-Verlag, 1985

[4] A.A. Il'ushin. *Plasticity*: "Gostehizdat", Moscow, 1948 (Russian)
[5] G.I. Marchuck, V.V. Shaidurov. *Difference schemes accuracy improvement*: "Nauka", Moskow, 1979 (Russian)
[6] N.I. Muskhelishvili. *Some main mathematical elasticity theory problems*: "Nauka", Moscow, 1966 (Russian)
[7] V.V. Shaidurov. *Multigrid finite element methods*: "Nauka", Moscow, 1989 (Russian)
[8] A.V. Trofimov. *Application of the Multigrid Iterative Methods to Stress Concentration Problems*: Dissertation for a Candidate degree in Physics and Mathematics, Dnepropetrovsk, 1996 (Russian)
[9] A.V. Trofimov. *Multigrid iterative algorithms for elasto-plastic boundary-value problems*: Deformation and Fracture in Solid Body Mechanics (special issue), Dnepropetrovsk, 1993, pp 102-113 (Russian)
[10] A.V. Trofimov, N.Y. Shvaiko. *Multigrid Methods for Flow Plasticity Theory Problems*: Deformation and Fracture in Solid Body Mechanics (special issue), Dnepropetrovsk, 1992, pp 75-83 (Russian)
[11] Umanski S.E. *Optimization for boundary-value problems solving confidant methods in mechanics*: "Naukova Dumka", Kiev, 1983 (Russian)